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APPLICATIONS OF $\alpha-\varphi$-GERAGHTY FOR THE EXISTENCE RESULTS IN A BOUNDARY VALUE PROBLEM WITH THE MITTAG-LEFFLER KERNEL

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Abstract. This research studies some important conditions for the existence of a solution for a boundary value problem with the Mittag-Leffler kernel. Our results are based on $\alpha-\varphi$-Geraghty type contractive mapping.

1. Introduction

Recently, the calculus of fractional derivatives has been considered an essential tool both in mathematics and in applications. That discussed and utilized in the modeling of many physical and chemical phenomena and engineering (see, for example, [1–7, 12, 13, 18, 19, 21, 22]).

The Caputo-Fabrizio is a type of derivative that was proposed by some authors in [8, 9, 15, 17].

An important subject that was presented by Atangana and Koca in the paper presented in Chaos Soliton and Fractal where they discussed the solution of equation ${}^{CF}_{t}D^{\alpha}_{0}y(t)=y$ and said that it is a special function but an exponential. To solve these defects, Atangana and Baleanu suggested a corrected version of a derivative without a singular kernel that satisfies the issues presented against that of Caputo-Fabrizio [10, 11]. This derivative is a well-known generalized Mittag-Leffler function.

In this study the following problem

$$(1.1)\qquad {}^{ABC}_{\varsigma}D^{\alpha}_{0}\eta(\varsigma)+a(\varsigma)f(t,\eta(\varsigma))=0,\qquad \varsigma\in[0,1],\ 1<\alpha\leq 2$$

$$(1.2)\qquad \eta'(0)=\eta(1)=0,$$

will be studied, where ${}^{ABC}_{\varsigma}D^{\alpha}_{0}$ is the Atangana-Baleano derivative in the sense of Caputo, $f:[0,1]\times\mathbf{R}^{+}\to\mathbf{R}^{+}$ is a continuous and $a:[0,1]\to\mathbb{R}$ is a continuous with $a(0)=0$. The sufficient conditions for the existence of the solution to the problem will be studied. This will be done by use of $\alpha-\varphi-$contraction mapping and a generalization of $\alpha-\varphi-$ Geragty contraction mapping introduced by Karapinar in [14].

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2. Preliminaries

Definition 2.1. Let $f \in H^1(a,b)$, the fractional derivative (Atangana-Baleano derivative in Caputo sense) It is expressed as:

$$ {}^{ABC}_{\varsigma}D^{\alpha}_{a}(f(\varsigma)) = \frac{B(\alpha)}{1-\alpha}\int_a^{\varsigma} f'(\varrho)E_{\alpha}\left[-\alpha\frac{(\varsigma-\varrho)^{\alpha}}{1-\alpha}\right]d\varrho, $$

where B is normalized function that has similarly properties with Caputo and Fabrizio suggested in their derivative.

Definition 2.2. The fractional integral associated with the Atangana-Baleano is defined by:

$$ {}^{AB}_{\varsigma}I^{\alpha}_{a}(f(\varsigma)) = \frac{1-\alpha}{B(\alpha)}f(\varsigma) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_a^{\varsigma}(\varsigma-\varrho)^{\alpha-1}f(\varrho)d\varrho. $$

Remark 2.3. As it is defined in [15], if $n \geq 1$, $\alpha \in [0,1]$ and $f^{(n)}(\varsigma) \in H^1(a,b)$ the fractional derivative ${}^{ABC}_{\varsigma}D^{\alpha+n}_{a}f(\varsigma)$ of order $\alpha+n$ can be defined by

$$ {}^{ABC}_{\varsigma}D^{\alpha+n}_{a}f(\varsigma) := {}^{ABC}_{\varsigma}D^{\alpha}_{a}D^{n}f(\varsigma). \tag{2.1} $$

Recently, Some generalization of the $\alpha-\varphi-$Geraghty contraction mapping (see [16].) were introduced in [14]. We recall some basic concepts and materials of this subject that we need to use in our proofs.
Let Ψ be the class of $\varphi : \mathbf{R}^+ \to \mathbf{R}^+$ with the following properties:

(a) φ is nondecreasing;
(b) $\varphi(\varrho+\varsigma) \leq \varphi(\varrho)+\varphi(\varsigma)$;
(c) φ is continuous;
(d) $\varphi(\varsigma) = 0 \Leftrightarrow \varsigma = 0$,

and let Φ be the class of $\phi : \mathbf{R}^+ \to \mathbf{R}^+$ with the following properties:

(i) ϕ is increasing,
(ii) $\forall\, x > 0, \phi(x) < x$,
(iii) $\beta(x) = \frac{\phi(x)}{x} \in \mathfrak{F}$,

where $\mathfrak{F}$ is a class of $\beta : \mathbf{R}^+ \to [0,1)$ with the following properties:

$$ \lim_{n\to\infty}\beta(\varsigma_n) = 1 \text{ implies } \lim_{n\to\infty}\varsigma_n = 0. $$

Including, $\phi(\varsigma) = \mu\varsigma$, with $0 \leq \mu < 1$, and $\phi(\varsigma) = \log(1+\varsigma)$ belong to Φ.

Definition 2.4. Assume that $T : X \to X$ and $\alpha : X \times X \to \mathbb{R}$. T is α-admissible if

$$ \alpha(\varsigma,\varrho) \geq 1 \Rightarrow \alpha(T\varsigma, T\varrho) \geq 1, $$

and T is triangular α-admissible if

$$ \alpha(\varsigma,\vartheta) \geq 1, \alpha(\vartheta,\varrho) \geq 1 \Rightarrow \alpha(\varsigma,\varrho) \geq 1. $$

Definition 2.5. [14] Let $\alpha : X \times X \to \mathbb{R}$ ((X,d) a metric space). $T : X \to X$ is said to be $\alpha-\varphi$-Geraghty contraction type if $\exists\, \beta \in \mathfrak{F}$ with the following properties:

$$ \alpha(\varsigma,\varrho)\varphi(d(T\varsigma,T\varrho)) \leq \beta(\varphi(d(\varsigma,\varrho)))\varphi(d(\varsigma,\varrho)), \tag{2.2} $$

where $\varphi \in \Psi$.

Theorem 2.6. *[20] Let $T : X \to X$ ((X, d) is a complete metric space) be an $\alpha-\varphi-$contractive and $\alpha-$admissible on X with $\alpha(\varsigma_0, T\varsigma_0) \geq 1$ for $\varsigma_0 \in X$. If ς_n is a sequence in X with $\alpha(\varsigma_n, \varsigma_{n+1}) \geq 1$ and $\varsigma_n \to \varsigma$ for some $\varsigma \in X$, Then $\alpha(\varsigma_n, \varsigma) \geq 1$. Then, T has a fixed point.*

Theorem 2.7. *[14] Let (X, d) be complete, $T : X \to X$ be a continuous $\alpha-\varphi-$Geraghty contractive and triangular $\alpha-$admissible with $\alpha(\varsigma_0, T\varsigma_0) \geq 1$ for some $\varsigma_0 \in X$. Then T has a fixed point $\varsigma^* \in X$, and $\{T^n\varsigma_0\} \to \varsigma^*$.*

3. The green function

Lemma 3.1. *The problem (1.1)-(1.2) is equivalent with the following equation:*

$$\eta(\varsigma) = \int_0^\varsigma G(\varsigma, \varrho)a(\varrho)f(\varrho, \eta(\varrho))d\varrho, \tag{3.1}$$

where

$$G(\varsigma, \varrho) = \begin{cases} -\dfrac{[(\varsigma-\varrho)^{\alpha-1} - (1-\varrho)^{\alpha-1}]}{B(\alpha-1)\Gamma(\alpha-1)}, & \varrho \leq \varsigma \\ \dfrac{(2-\alpha)}{B(\alpha-1)} + \dfrac{(1-\varrho)^{\alpha-1}}{B(\alpha-1)\Gamma(\alpha-1)}, & \varrho \geq \varsigma. \end{cases} \tag{3.2}$$

Proof. Let $\beta = \alpha - 1$, then equation (1.1) changes to the equation

$${}^{ABC}_{\varsigma}D_0^{\beta+1}\eta(\varsigma) + a(\varsigma)f(\varsigma, \eta(\varsigma)) = 0. \tag{3.3}$$

We consider the boundary value problem

$${}^{ABC}_{\varsigma}D_0^{\beta+1}\eta(\varsigma) + g(\varsigma) = 0, \quad \varsigma \in [0, 1], \tag{3.4}$$

$$\eta(0)' = \eta(1) = 0, \tag{3.5}$$

where $g : [0, 1] \to \mathbb{R}$ with $g(0) = 0$. Appling the Atangana-Baleanu integral of order β on both side of (3.4) we get

$$\eta'(\varsigma) = -\frac{1-\beta}{B(\beta)}g(\varsigma) - \frac{\beta}{B(\beta)\Gamma(\beta)}\int_0^\varsigma (\varsigma-\varrho)^{\beta-1}g(\varrho)d\varrho + C_1. \tag{3.6}$$

By integrating we obtain

$$\eta(\varsigma) = -\frac{1-\beta}{B(\beta)}\int_0^\varsigma g(\varrho)d\varrho - \frac{\beta}{B(\beta)}\frac{1}{\Gamma(\beta+1)}\int_a^\varsigma (\varsigma-\varrho)^{\beta}g(\varrho)d\varrho + C_1\varsigma + C_2, \tag{3.7}$$

or

$$\eta(\varsigma) = -\frac{2-\alpha}{B(\alpha-1)}\int_0^\varsigma g(\varrho)d\varrho - \frac{\alpha-1}{B(\alpha-1)}\frac{1}{\Gamma(\alpha)}\int_0^\varsigma (\varsigma-\varrho)^{\alpha-1}g(\varrho)d\varrho + C_1\varsigma + C_2. \tag{3.8}$$

From the first boundary condition and the property $g(0) = 0$ we have

$$\eta'(0) = C_1 = 0 \Rightarrow C_1 = 0.$$

Appling the second boundary condition on equation (3.8) we get

$$\eta(1) = -\frac{2-\alpha}{B(\alpha-1)}\int_0^1 g(\varrho) - \frac{\alpha-1}{B(\alpha-1)\Gamma(\alpha)}\int_0^1 (1-\varrho)^{\alpha-1}g(\varrho)d\varrho + C_2 = 0.$$

So

$$C_2 = \frac{2-\alpha}{B(\alpha-1)}\int_0^1 g(\varrho)d\varrho + \frac{\alpha-1}{B(\alpha-1)\Gamma(\alpha)}\int_0^1 (1-\varrho)^{\alpha-1}g(\varrho)d\varrho.$$

Thus we have

$$\begin{aligned}\eta(\varsigma) &= -\frac{2-\alpha}{B(\alpha-1)}\int_0^\varsigma g(\varrho)d\varrho - \frac{\alpha-1}{B(\alpha-1)}\frac{1}{\Gamma(\alpha)}\int_0^\varsigma(\varsigma-\varrho)^{\alpha-1}g(\varrho)d\varrho \\ &\quad +\frac{(2-\alpha)}{B(\alpha-1)}\int_0^1 g(\varrho)d\varrho + \frac{(\alpha-1)}{B(\alpha-1)\Gamma(\alpha)}\int_0^1(1-\varrho)^{\alpha-1}g(\varrho)d\varrho \\ &= -\frac{2-\alpha}{B(\alpha-1)}\int_0^\varsigma g(\varrho)d\varrho - \frac{1}{B(\alpha-1)\Gamma(\alpha-1)}\int_0^\varsigma(\varsigma-\varrho)^{\alpha-1}d\varrho \\ &\quad +\frac{(2-\alpha)}{B(\alpha-1)}\int_0^1 g(\varrho)d\varrho + \frac{1}{B(\alpha-1)\Gamma(\alpha-1)}\int_0^1(1-\varrho)^{\alpha-1}d\varrho \\ &= \int_0^1 G(\varsigma,\varrho)g(\varrho)d\varrho.\end{aligned}$$

□

Lemma 3.2. *For the mentioned* $G(\varsigma,\varrho)$*, the following conditions hold*

(1) $G(\varsigma,\varrho)\geq 0$ *for* $\varsigma,\varrho\in[0,1]$,

(2) $G(\varsigma,\varrho)\leq \frac{2-\alpha}{B(\alpha-1)}+\frac{(1-\varrho)^{\alpha-1}}{\Gamma(\alpha-1)B(\alpha-1)}$ *for all* $\varsigma,\varrho\in[0,1]$,

(3) $G(\varsigma,\varrho)\geq (1-\varsigma^{\alpha-1})\frac{(1-\varrho)^{\alpha-1}}{\Gamma(\alpha-1)B(\alpha-1)}$ *for all* $\varsigma,\varrho\in[0,1]$.

Proof. (1) For $\varrho\leq\varsigma$ we get $(1-\varrho)^{\alpha-1}-(\varsigma-\varrho)^{\alpha-1}\geq 0$, So $G(\varsigma,\varrho)\geq 0$. Moreover clearly, for $\varrho\geq\varsigma$, $G(\varsigma,\varrho)\geq 0$.

(2) Let $\varrho\leq\varsigma$, then $\frac{\partial G}{\partial\varsigma}=\frac{-(\alpha-1)(\varsigma-\varrho)^{\alpha-2}}{\Gamma(\alpha-1)B(\alpha-1)}=0$ implies $\varrho=\varsigma$, so $G(\varsigma,\varrho)\leq \frac{(1-\varrho)^{\alpha-1}}{\Gamma(\alpha-1)B(\alpha-1)}$. Moreover clearly, for $\varrho\geq\varsigma$ we obtain $G(\varsigma,\varrho)\leq\frac{2-\alpha}{B(\alpha-1)}+\frac{(1-\varrho)^{\alpha-1}}{\Gamma(\alpha-1)B(\alpha-1)}$. Hence $G(\varsigma,\varrho)\leq\frac{2-\alpha}{B(\alpha-1)}+\frac{(1-\varrho)^{\alpha-1}}{\Gamma(\alpha-1)B(\alpha-1)}$.

(3) For $\varrho\leq\varsigma$ we get

$$\begin{aligned}G(\varsigma,\varrho) &= \frac{(1-\varrho)^{\alpha-1}-(\varsigma-\varrho)^{\alpha-1}}{B(\alpha-1)\Gamma(\alpha-1)}\geq\frac{(1-\varrho)^{\alpha-1}-\varsigma^{\alpha-1}(1-\varrho)^{\alpha-1}}{B(\alpha-1)\Gamma(\alpha-1)} \\ &= (1-\varsigma)^{\alpha-1}\frac{(1-\varrho)^{\alpha-1}}{B(\alpha-1)\Gamma(\alpha-1)}.\end{aligned}$$

Moreover for $s\geq\varsigma$ we obtain

$$\begin{aligned}G(\varsigma,\varrho) &= \frac{2-\alpha}{B(\alpha-1)}+\frac{(1-\varrho)^{\alpha-1}}{\Gamma(\alpha-1)B(\alpha-1)} \\ &\geq \frac{(1-\varrho)^{\alpha-1}}{\Gamma(\alpha-1)B(\alpha-1)} \\ &\geq (1-\varsigma)^{\alpha-1}\frac{(1-\varrho)^{\alpha-1}}{\Gamma(\alpha-1)B(\alpha-1)}\end{aligned}$$

□

4. **Existence Results**

Consider the following hypotheses:

H0: Considering the cone $P=\{\eta(\varsigma):\eta(\varsigma)\in\mathbb{R}^+,\ \varsigma\in[0,1]\}$;

H1: $\exists\, \xi : \mathbb{R}^2 \to \mathbb{R}^+$ and $\varphi \in \Psi$ with

$$|a(\varsigma)f(\varsigma, a) - a(\varsigma)f(\varsigma, b)| \leq \lambda\varphi(|a - b|),$$

for $0 \leq \varsigma \leq 1$ and $a, b \in \mathbb{R}^+$ with $\xi(a, b) \geq 0$, where $\lambda < \frac{B(1-\alpha)\alpha\Gamma(\alpha-1)}{(2-\alpha)\Gamma(\alpha-1)+1}$;

H2: $\exists\, \eta_0 \in C([0, 1])$ with $\xi\left(\eta_0(\varsigma), \int_0^1 G(\varsigma, \varrho)a(\varrho)f(\varrho, \eta_0(\varrho))d\varrho\right) \geq 0$, $\varsigma \in [0, 1]$;

H3: for $\varsigma \in [0, 1]$ and $\eta, \zeta \in C([0, 1]), \xi(\eta(\varsigma), \zeta(\varsigma)) \geq 0$ implies $\xi\left(\int_0^1 G(\varsigma, \varrho)a(\varrho)f(\varrho, \eta(\varrho))d\varrho, \int_0^1 G(\varsigma, \varrho)a(\varrho)f(\varrho, \zeta(\varrho))d\varrho\right) \geq 0$;

H4: if $\{\eta_n\}$ is a sequence in $C([0, 1])$ with $\eta_n \to \eta$ and $\xi(\eta_n, \eta_{n+1}) \geq 0$, then $\xi(\eta_n, \eta) \geq 0$;

Theorem 4.1. *Assume that* ***H1-H4*** *hold. Then,* (1.1)-(1.2) *has at least one solution.*

Proof. We define $T : P \to P$ by

$$T\eta(\varsigma) = \int_0^1 G(\varsigma, \varrho)a(\varrho)f(\varrho, \eta(\varrho))d\varrho. \tag{4.1}$$

By using Lemma 3.1 $\eta \in C([0, 1])$ is a solution of (1.1)-(1.2) if $\eta \in C([0, 1])$ is a fixed point of the operator (4.1). Now let $\eta, \zeta \in C([0, 1])$ with $\xi(\eta(\varsigma), \zeta(\varsigma)) \geq 0$ for $\varsigma \in [0, 1]$. By using **H1**, we have

$$\begin{aligned}
|T\eta(\varsigma) - T\zeta(\varsigma)| &\leq \left|\int_0^1 G(\varsigma, \varrho)a(\varrho)[f(\varrho, \eta(\varrho)) - f(\varrho, \zeta(\varrho))]d\varrho\right| \\
&\leq \lambda \int_0^1 |G(\varsigma, \varrho)|\varphi(|\eta(\varrho) - \zeta(\varrho)|)d\varrho \\
&\leq \lambda\varphi(\|\eta - \zeta\|_\infty)\frac{(2 - \alpha)\Gamma(\alpha - 1) + 1}{B(1 - \alpha)\alpha\Gamma(\alpha - 1)} \\
&\leq \varphi(\|\eta - \zeta\|_\infty).
\end{aligned}$$

Thus, for each $\eta, \zeta \in C([0, 1])$ with $\xi(\eta(\varsigma) - \zeta(\varsigma)) \geq 0$ for all $\varsigma \in [0, 1]$, we have

$$\|T\eta - T\zeta\|_\infty \leq \varphi(\|\eta - \zeta\|_\infty).$$

We define $\alpha : C([0, 1]) \times C([0, 1]) \to \mathbf{R}^+$ by

$$\alpha(\eta, \zeta) = \begin{cases} 1 & \xi(\eta(\varsigma), \zeta(\varsigma)) \geq 0 \text{ for all } \varsigma \in [0, 1] \\ 0 & \text{otherwise.} \end{cases} \tag{4.2}$$

Hence, $\alpha(\eta, \zeta)d(T\eta, T\zeta) \leq \varphi(d(\eta, \zeta))$ for all $\eta, \zeta \in C([0, 1])$. So F is an $\alpha-\varphi-$contractive mapping. By **H3**, we obtain

$$\alpha(\eta, \zeta) \geq 1 \Rightarrow \varphi(\eta(\varsigma), \zeta(\varsigma)) \geq 0 \Rightarrow \varphi(T\eta(\varsigma), T\zeta(\varsigma)) \geq 0 \Rightarrow \alpha(T\eta, T\zeta) \geq 1,$$

for $\eta, \zeta \in C([0, 1])$. Hence, T is $\alpha-$admissible. From **H2**, there exists $\eta_0 \in C([0, 1])$ with $\alpha(\eta_0, T\eta_0) \geq 1$.
Finally, from **H4** and using Theorem 2.6, we deduce the existence of $\eta^* \in C([0, 1])$ with $\eta^* = T\eta^*$. Hence, u^* is a solution. □

Our other result obtain from Theorem 2.7. First consider the condition.

H′1: $\exists\, \xi : \mathbb{R}^2 \to \mathbb{R}$ with $\xi(a, b) \geq 0$ such that

$$|f(\varsigma, a) - f(\varsigma, b)| \leq \lambda \log(|a - b| + 1), \text{ where, } \lambda < \frac{B(1 - \alpha)\alpha\Gamma(\alpha - 1)}{(2 - \alpha)\Gamma(\alpha - 1) + 1};$$

Theorem 4.2. *Assume $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous and **H′1**, **H2-H4** hold, then the problem* (1.1)-(1.2) *has at least one solution.*

Proof. We consider the operator (4.1) again. Let $\eta, \zeta \in C([0,1])$ such that $\xi(\eta(\varsigma), \zeta(\varsigma)) \geq 0$ for $\varsigma \in [0,1]$, we get

$$\begin{aligned} d(T\eta, T\zeta) &\leq \Big| \int_0^1 G(\varsigma, \varrho) a(\varrho)[f(\varrho, \eta(\varrho)) - f(\varrho, \zeta(\varrho))] d\varrho \Big| \\ &\leq \Big| \int_0^1 G(\varsigma, \varrho) \lambda \log(|\eta(\varrho) - \zeta(\varrho)| + 1) d\varrho \Big| \\ &\leq \log(\|\eta - \zeta\|_\infty + 1) \lambda \int_0^1 |G(\varsigma, \varrho) d\varrho| \\ &\leq \lambda \log(\|\eta - \zeta\|_\infty + 1) \left(\frac{B(1-\alpha)\alpha\Gamma(\alpha-1)}{(2-\alpha)\Gamma(\alpha-1)+1} \right) \\ &\leq \log(\|\eta - \zeta\|_\infty + 1) = \log(d(\eta, \zeta) + 1), \end{aligned}$$

which yields that

$$\log(d(T\eta, T\zeta)+1) \leq \log(\log(d(\eta,\zeta)+1)+1) = \frac{\log(\log(d(\eta,\zeta)+1)+1)}{\log(d(\eta,\zeta)+1)} \log(d(\eta,\zeta)+1)$$

Let $\varphi(x) = \log(x+1)$ and $\beta(x) = \frac{\varphi(x)}{x}$. It is clear, $\varphi : \mathbf{R}^+ \to \mathbf{R}^+$ is subadditive, nondecreasing, continuous function. In addition φ is positive in $(0,\infty)$ and $\varphi(0) = 0$. Morever $\varphi(x) < x$ for any $\beta \in \mathfrak{F}$.
Thus for all $\eta, \zeta \in C([0,1])$ with $\xi(\eta(\varsigma), \zeta(\varsigma)) \geq 0$ with $\xi(\eta(\varsigma), \zeta(\varsigma)) \geq 0$ for $\varsigma \in [0,1]$, we get $\varphi(d(T\eta, T\zeta)) < \beta(\varphi(d(\eta,\zeta)))\varphi(d(\eta,\zeta))$.
We consider the function (4.2). Then for $\eta, \zeta \in C([0,1])$, we get

$$\alpha(\eta,\zeta) d(T\eta, T\zeta) < \beta(d(\eta,\zeta)) d(\eta,\zeta).$$

Obviously, $\alpha(\eta,\zeta) = 1$ and $\alpha(\zeta, w) = 1$ implies $\alpha(\eta, w) = 1$ for $\eta, \zeta, w \in C([0,1])$.
If $\alpha(\eta,\zeta) = 1$ for all $\eta, \zeta \in C([0,1])$, then $\xi(\eta(\varsigma), \zeta(\varsigma)) \geq 0$. From (**H3**) we get $\xi(T\eta(\varsigma), T\zeta(\varsigma)) \geq 0$, and so $\alpha(T\eta, T\zeta) = 1$. Thus T is triangular admissible.
From (**H2**) there exists $\eta_0 \in C([0,1])$ such that $\alpha(\eta_0, T\eta_0) = 1$.
By (**H5**), we find that, for point x of sequence $\{\eta_n\} \in C([0,1])$ with $\alpha(\eta_n, \eta_{n+1}) = 1$, $\lim_{n\to\infty} \alpha(\eta_n, \eta) = 1$.
By applying theorem 2.7, T has a fixed point in $C([0,1])$ and this is solution of (1.1).

□

Example 4.3. Let $\varphi(r) = r$, $\xi(x,z) = xz$,
$\eta_n(\varsigma) = \dfrac{\varsigma}{n^2+1}$. Consider $f : I \times C([0,1]) \to [-1,1]$ and the bondary value problem

$$\frac{D^{\frac{3}{2}}}{D\varsigma}\eta(\varsigma) + \varsigma f(\varsigma, \eta(\varsigma)) = 0, \tag{4.3}$$

where $f(\varsigma, \eta(\varsigma)) = \sin \eta(\varsigma)$, $\varsigma \in I$, also,

$$\eta'(0) = \zeta(1) = 0,$$

So

$$|\varsigma f(\varsigma, \eta(\varsigma)) - \varsigma f(\varsigma, \zeta(\varsigma))| \leq \lambda |\sin \eta(\varsigma) - \sin \zeta(\varsigma)| \leq \lambda |\eta(\varsigma) - \zeta(\varsigma)|,$$

when $\varsigma \in I$ and $\eta(\varsigma), \zeta(\varsigma) \in [-1,1]$ with $\xi(\eta(\varsigma), \zeta(\varsigma)) \geq 0$. If $\eta_0(\varsigma) = \varsigma$, then

$$\xi\big(\eta_0(\varrho), \int_0^1 G(\varsigma,\varrho)\varsigma f(\varsigma, \eta_0(\varrho))d\varrho\big) \geq 0.$$

for $\varrho \in I$. Also,
$\xi(y(\varrho), z(\varrho)) = y(\varrho)z(\varrho) \geq 0$ implies that

$$\xi(\int_0^1 G(\varsigma,\varrho)\varsigma f(\varrho, y_(\varrho))d\varrho, \int_0^1 G(\varsigma,\varrho)\varsigma f(\varrho, z(\varrho))d\varrho) \geq 0;$$

It is obviously that condition (H4) in Theorem (4.1) hold. hence, the all of conditions Theorem (4.1) satisfied. So from Theorem (4.1) the problem (1.1)-(1.2) has at least one solution.

References

1. Afshari H.; Karapinar, E. A solution of the fractional differential equations in the setting of b-metric space. Carpathian Math. Publ. 13 (2021), no. 3, 764–774.
2. Afshari H., Roomi V., Kalantari S., Existence of solutions of some boundary value problems with impulsive conditions and ABC-fractional order, Filomat 37:11 (2023), 3639–3648 https://doi.org/10.2298/FIL2311639A.
3. Afshari H., Roomi V., Kalantari S., The existence of solutions of the inclusion problems involving Caputo and Hadamard fractional derivatives by applying some new contractions, Journal of Nonlinear and Convex Analysis 23(6) (2022) 1213-1229.
4. Afshari H., Karapinar E., A discussion on the existence of positive solutions of the boundary value problems via φ-Hilfer fractional derivative on b-metric spaces. Adv. Difference Equ. 2020, 616. 13 (2021), no. 3, 764-774.
5. Agarwal R P. Formulation of Euler-Lagrange equations for fractional variational problems. J Math Anal Appl. 2002 272; 368-379.
6. Ahmadkhanlu A, Jahanshahi M. On the Existence and Uniqueness of Solution of Initial Value problem for Fractional order Differential Equations on Time scales. *Bull. Ira. Math. Soc.* 2012; **38**(1): 241-252.
7. A. Ahmadkhanlu, *Existence and uniqueness results for a class of fractional differential equations with an integral fractional boundary condition*, Filomat, 31 (2017), 1241–1249.
8. Atangana A, Badr S. Analysis of the Keller–Segel model with a fractional derivative without singular kernel. Entropy 2015; 17 4439–4453.
9. Atangana A, Badr S. Extension of the RLC electrical circuit to fractional derivative without singular kernel. Adv Mech Eng. 2105 7; 1–6.
10. Atangana A, Baleanu D. New fractional derivatives with nonlocal and non-singular kernel: Theory and applica- tion to heat transfer model. Therm Sci. 2016 20; 763-769.
11. Atangana A, Koca I. Chaos in a simple nonlinear system with Atangana-Baleanu derivatives with fractional order. Chaos Solitons Fractals 2016 89; 447-454
12. Benchohra, M.; Karapınar, E.; Lazreg, J.E.; Salim, A. Advanced Topics in Fractional Differential Equations: A Fixed Point Approach; Springer: Cham, Switzerland, 2023.
13. Benchohra, M.; Karapınar, E.; Lazreg, J.E.; Salim, A. Fractional Differential Equations: New Advancements for Generalized Fractional Derivatives; Springer: Cham, Switzerland, 2023.
14. Karapinar E., $\alpha-\varphi$-Geraghty contraction type mappings and some related fixed point results. Filomat. 2014 28(1); 37–48.
15. Caputo M, Fabrizio M, A. new definition of fractional derivative without singular kernel. Progr Fract Differ Appl 2015 1; 73–85.
16. Geraghty M. On contraction mappings. Pros Amer Math Soc. 1973 40; 604–608.
17. Losada J, Nieto J J. Properties of the new fractional derivative without singular kernel. Progr Fract Differ Appl 2015 1; 87–92.
18. Miller K S, Ross B., An Introduction to the Fractional Calculus and Fractional Differential Equation. New York, USA: Wiley, 1993.
19. Oldham K. B, Spanier J, The Fractional Calculus. New York, USA: Academic Press, 1974.
20. Samet B, Vetro C, Vetro P. Fixed point theorems for $\alpha-\varphi-$contractive type mappings. Nonlinear Anal Theor. 2012 75; 2154–2165.

21. Samko S G, Kilbas A A, Marichev O I. *Fractional Integral and Derivative: Theory and Applications*. Gordon & Breach, Yverdon, 1993.
22. Podlubny I. *Fractional Differential Equations*. New York, USA: Academic Press, 1999.

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