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# Bounded Inverse-Slashed Pareto Model: Structural Properties and Real-Life Applications

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**Abstract**

A novel probability model with bounded support is introduced. The formulation of this new probability model is based on inverting the Slashed Pareto distribution. This new distribution has the merit of being very simple and not involving any complex mathematical function in its construction. Some interesting properties like moments, skewness and kurtosis, unimodality, L-Moments, L-skewness and L-kurtosis would be explored in detail. Various Survival properties including survival function, hazard rate function and mean residual life(MRL) like have been given. For estimating the parameters contained in the new model, methods like Method of Moments (MOM) and Maximum Likelihood Estimation (MLE) have been used.

*Keywords:* Probability model Moments; Distribution Maximum Likelihood Estimation.
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## 1. Introduction

Although among the many alternatives and generalizations [2], It's fair to say that beta distributions represent a major family of continuous distributions with support defined on $(0, 1)$. The probability density

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function (pdf) of beta distribution with parameters $m > 0$ and $n > 0$ is given by:

$$g(x) = \frac{1}{B(m,n)} x^{m-1}(1-x)^{n-1}, \qquad 0 < x < 1, \tag{1}$$

with $B(\cdot,\cdot)$ is the beta function. The beta distribution is reasonably good in many ways but with shortcomings like its cumulative distribution function (CDF) is an incomplete beta function and hence as a consequence its quantile function as well. In addition to beta distribution, so many new models having bounded support on unit interval have been studied in the literature. Jones [7] Kumaraswamy distribution, the McDonalds generalized beta distribution [8], Gordy's confluent hypergeometric distribution [4], the Gauss hypergeometric distribution discussed by Armero and Bayarri [3], the transformed gamma distribution [6] and the LogLindley distribution proposed by Gómez et al. [5] are recent developments in this field of literature. Among all the models involve special functions in their construction except Kumaraswamy and LogLindley distribution.

Keeping the above shortcomings in mind, a new two-parameter distribution with bounded domain is being proposed here that can be considered another choice to beta, Kumarswamy and Log-Lindley distribution. Besides involving only two parameters, it has benefit over other models as it doesn't contain any special functions. This distribution is obtained by taking the inverse of a random variable (rv) following slashed Pareto distribution.

The rest of the paper is structured as: The Section 2 explored the derivation of proposed distribution along with unimodality and its nested models. Distributional properties like distribution function, survival function, hazard rate function etc have been presented in Section 3 followed by discussion on order statistics in Section 4. Moments and other associated properties have been discussed in Section 5. For estimation of parameters of the model, MOM and MLE have been presented in section 6. The statistical stability of the proposed model were checked by Monte Carlo simulation process in Section 7. In the penultimate section, the numerical illustration were presented. Lastly, Section 9 ends with work some remarks.

## 2. Inverse-Slashed Pareto Distribution

A rv Y is said to follow Inverse-Slashed Pareto distribution if its pdf is given by

$$f(y;\alpha,q) = \frac{\alpha q}{q-\alpha}\left(y^{\alpha-1} - y^{q-1}\right), \quad 0 < y < 1, \alpha > 0, q > \alpha. \tag{2}$$

Henceforth, a rv following pdf (2) will be denoted by $Y \sim \mathbb{ISP}(\alpha, q)$. It is pertinent to note mention here that for $\alpha > q$ it has the same density function, that is called the problem of identifiability. For removing this problem we consider $q > \alpha$. The pdf (2) can be obtained by taking inverse of Slashed Pareto distribution which has been derived in Theorem 2.1.

**Theorem 2.1.** *If a rv $U \sim U(0,1)$, $X$ is another rv following Pareto distribution with the notation $X \sim Par(\alpha, 1)$ and $Z = \frac{X}{U^{\frac{1}{q}}}$ is a Slashed Pareto random variate, Then rv $Y = \frac{1}{Z}$ is an Inverse Slashed Pareto distribution whose pdf is given in (2).*

*Proof.* By taking into account that the cumulative distribution function (cdf) $F_X(z) = \int_{(\frac{1}{z})^q}^{1} \left[1 - (\frac{1}{zu^{\frac{1}{q}}})^{\alpha}\right] dx$, then using Jacobian transformation method such that $|J| = \frac{1}{y^2}$, the result follows after some computations. □

The pdf (2) of the proposed model can be derived analytically as well.

Let $f_1(y)$ and $f_2(y)$ be two integrable functions over support [0,1] such that $f_1(y) > f_2(y)\ \forall \quad y \in [0,1]$, then $f_1(y) - f_2(y)$ is also integrable. Moreover if $\int_0^1 (f_1(y) - f_2(y))dy = k$ then $\frac{f_1(y)-f_2(y)}{k}$ is always a density over support [0,1]. Therefore, choosing $f_1(y) = y^{\alpha-1}$ and $f_2(y) = y^{q-1}$, $q > \alpha$ such that $k = \frac{q-\alpha}{\alpha q}$ and after doing

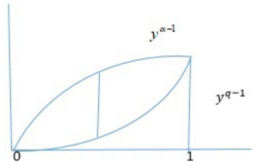


Figure 1: Analytical Genesis

some easy mathematical simplifications, Inverse-Slashed Pareto distribution (2) can be derived analytically. The first derivative of pdf (2) is:

$$\frac{d}{dy}f(y;\alpha,q)=\frac{\alpha q}{q-\alpha}\left[(\alpha-1)y^{\alpha-2}-(q-1)y^{q-2}\right]$$

this means that the pdf (2) is unimodal with maximum at:

- If $0<\alpha<1$, $0<q<1$ & always $q>\alpha$, then

$$y_{max}=\left[\frac{\alpha-1}{q-1}\right]^{\frac{1}{q-\alpha}}.$$

- If $\alpha>1$, $q>1$ & always $q>\alpha$, then

$$y_{max}=\left[\frac{\alpha-1}{q-1}\right]^{\frac{1}{q-\alpha}}.$$

- If $0<\alpha<1$ and $q>1$ and $q>\alpha$, then $y_{max}=0$.

Figure 2 shows the pdf plot of proposed model for different choices parameters $\alpha$ and $q$.

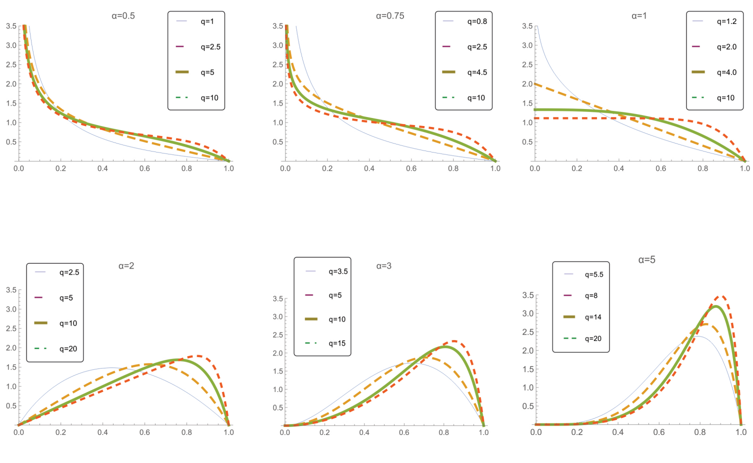


Figure 2: pdf plot of $\mathbb{ISP}(\alpha,q)$ distribution for different values of $\alpha$ and $q$.

**Theorem 2.2.** *If $X \sim \mathbb{ISP}(\alpha, q)$, then $Y = X^{\theta} \sim \mathbb{ISP}(\frac{\alpha}{\theta}, \frac{q}{\theta})$.*

*Proof.* It is straightforward to prove this result. □

In the following subsection, we will show some existing distributions appears to be particular cases of $\mathbb{ISP}(\alpha, q)$ by choosing suitable value for parameters of the model.

### *2.1. Nested Models of $\mathbb{ISP}(\alpha, q)$ Distribution*

1. For $\alpha = 1$ in pdf (2), then we get a new density whose pdf is:

$$f(y; q) = \frac{q}{q-1}\left(1 - y^{q-1}\right), \quad q > 0, \quad 0 < y < 1.$$

2. For $q \to \alpha$ in pdf (2), and after doing some easy simplification then we get a new density function as:

$$f(y; \alpha) = -\alpha^2\left(y^{\alpha-1}\right)\log(y), \quad \alpha > 0, \quad 0 < y < 1.$$

3. For $q \to \infty$ in pdf (2), the new density function is obtained with the pdf as:

$$f(y, \alpha) = \alpha y^{\alpha-1}, \quad \alpha > 0, \quad 0 < y < 1.$$

## 3. Distributional Properties

1. The CDF of the $\mathbb{ISP}(\alpha, q)$ is given by

$$F_Y(y; \alpha, q) = \frac{qy^{\alpha} - \alpha y^{q}}{q - \alpha}, \quad 0 < y < 1, \quad \alpha > 0, \quad q > \alpha. \tag{3}$$

2. The Survival function of the $\mathbb{ISP}(\alpha, q)$ is given by

$$S_Y(y; \alpha, q) = 1 - \frac{qy^{\alpha} - \alpha y^{q}}{q - \alpha}, \quad 0 < y < 1, \quad \alpha > 0, \quad q > \alpha. \tag{4}$$

3. The Hazard Function of the $\mathbb{ISP}(\alpha, q)$ is given by

$$\begin{aligned} h(y; \alpha, q) =& \frac{f(y; \alpha, q)}{S_Y(y; \alpha, q)} \\ =& \frac{\alpha q\left[y^{\alpha} - y^{q}\right]}{y\left[\alpha\left(y^{q} - 1\right) + q\left(1 - y^{q}\right)\right]}, \quad 0 < y < 1, \quad \alpha > 0, \quad q > \alpha. \end{aligned} \tag{5}$$

   The behavior of Survival function and Hazard function have been illustrated in Figure 3 and 4 , respectively with different combinations of parameters.

4. For a non-negative continuous rv $Y$ the MRL function is defined as $\mu(y) = E(Y - y|Y > y)$ and is calculated by:

$$\mu(y) = \frac{1}{S(y)}\int_{y}^{\infty} S(y)dy.$$

   Considering $S(y) = S(y|\alpha, q) = 1 - \frac{qy^{\alpha} - \alpha y^{q}}{q - \alpha}$, the survival function of the $\mathbb{ISP}(\alpha, q)$ distribution, we have

$$\mu(y|\alpha, q) = \frac{\frac{\alpha(q-\alpha)\left(-y^{q+1} + q(y-1) + y\right)}{(q+1)(\alpha + q(y^{\alpha}-1) - \alpha y^{q})} - y}{\alpha + 1}.$$

   Note that $\lim_{y \to 0} \mu(y|\alpha, q) = \frac{\alpha q}{(q+1)(\alpha+1)}$ and while as $\lim_{y \to 1} \mu(y|\alpha, q) = 0$.

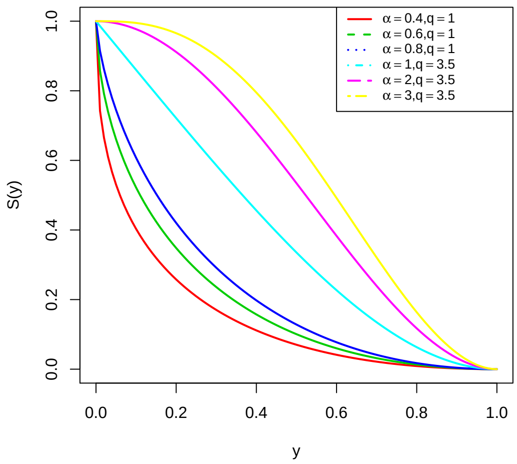


Figure 3: Plot of Survival function for different choices of parameters.

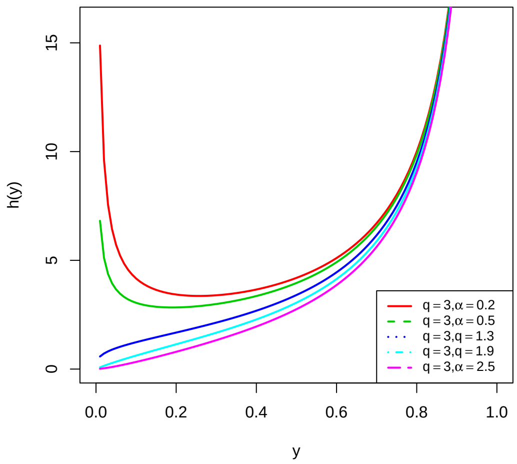


Figure 4: Plot of Hazard function for different choices of parameters.

**Theorem 3.1.** *The weighted version of* $\mathbb{ISP}(\alpha, q)$ *distribution with weight function* $\mathbb{W}(y) = Y^s$, $s > 0$ *is given by*

$$f_S(y) = \frac{y^s(\alpha+s)(q+s)}{q-\alpha}(y^{\alpha-1} - y^{q-1}), \quad 0 < y < 1, -\infty < s < \infty. \tag{6}$$

*Proof.* Since $Y \sim \mathbb{ISP}(\alpha, q)$, and it is given that $\mathbb{W}(y) = Y^s$. By using the definition of weighted class of distributions ([9]), the weighted pdf is

$$f_S(y) = \frac{\mathbb{W}(y)f(y)}{[E(\mathbb{W}(y))]}. \tag{7}$$

Now,

$$E[\mathbb{W}(y)] = E[Y^s] = \frac{\alpha q}{(q+s)(\alpha+s)}. \tag{8}$$

The required result can be obtained easily by substituting (8) and (2) in (7). □

## 4. Order Statistics

Consider a sequence $Y_1, Y_2, \ldots, Y_n$ of $n$ independent and identically distributed rv's, each with CDF $F(y)$. The pdf of largest order statistics $Y_{(n)}$ is given by:

$$\begin{aligned} f_n(y) =& n\,[F(y)]^{n-1} f(y) \\ =& \frac{\alpha n q\,(y^\alpha - y^q)\left(\frac{q y^\alpha - \alpha y^q}{q-\alpha}\right)^{n-1}}{y(q-\alpha)}, \quad 0 < y < 1, \quad \alpha > 0, \quad q > \alpha. \end{aligned} \tag{9}$$

Also, the pdf of smallest order statistics $Y_{(1)}$ is given by:

$$\begin{aligned} f_1(y) =& n\,[1-F(y)]^{n-1} f(y) \\ =& \frac{\alpha n q\,(y^\alpha - y^q)\left(\frac{\alpha y^q - q y^\alpha}{q-\alpha}+1\right)^{n-1}}{y(q-\alpha)}, \quad 0 < y < 1, \quad \alpha > 0, \quad q > \alpha. \end{aligned} \tag{10}$$

## 5. Moments and other Associated Properties

### *5.1. Raw Moments*

The $r^{th}$ moment $\mathbb{ISP}(\alpha, q)$ about origin is:

$$E(Y^r) = \frac{\alpha q}{(q+r)(\alpha+r)}. \tag{11}$$

In particular, the first four raw moments of $\mathbb{ISP}(\alpha, q)$ can be obtained easily by putting $r = 1, 2, 3, 4$ in (11) and are as follows:

$$\begin{aligned} \mu_1^{'} =& \frac{\alpha q}{(q+1)(\alpha+1)}, \\ \mu_2^{'} =& \frac{\alpha q}{(q+2)(\alpha+2)}, \\ \mu_3^{'} =& \frac{\alpha q}{(q+3)(\alpha+3)}, \\ \mu_4^{'} =& \frac{\alpha q}{(q+4)(\alpha+4)}, \end{aligned}$$

The variance $(\mu_2)$ is obtained as:

$$\begin{aligned} \mu_2 =& \mu_2^{'} - (\mu_1^{'})^2 \\ =& \alpha q\left[\frac{1}{(\alpha+2)(q+2)} - \frac{\alpha q}{(\alpha+\alpha q+q+1)^2}\right]. \end{aligned} \tag{12}$$

*5.2. Coefficient of Variation*

Coefficient of Variation (C.V.) ($\frac{\sigma}{\bar{x}}$) is:

$$C.V. = \frac{(\alpha+1)(q+1)\sqrt{\alpha q\left(\frac{1}{(\alpha+2)(q+2)} - \frac{\alpha q}{(\alpha+\alpha q+q+1)^2}\right)}}{\alpha q}.$$

The table (1) contains the information about C.V. for different combinations of parameters.

Table 1: Numerical values of C.V. for different choices of parameters $\alpha$ and $q$

| q ↓ | $\alpha = 1$ | $\alpha = 2.5$ | $\alpha = 4$ | $\alpha = 6$ | $\alpha = 8$ | $\alpha = 10$ | $\alpha = 12$ | $\alpha = 14$ | $\alpha = 16$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.881917 | 0.672199 | 0.6236 | 0.60092 | 0.5916 | 0.1856 | 0.1543 | 0.1432 | 0.12671 |
| 2.5 | 0.672199 | 0.430905 | 0.366414 | 0.334027 | 0.3302 | 0.31299 | 0.3089 | 0.2967 | 0.27539 |
| 4 | 0.62361 | 0.366414 | 0.29167 | 0.25173 | 0.1654 | 0.2244 | 0.0827 | 0.07765 | 0.0057 |
| 6 | 0.600925 | 0.334027 | 0.2517 | 0.20518 | 0.0811 | 0.1712 | 0.1160 | 0.1023 | 0.1006 |
| 8 | 0.591608 | 0.320156 | 0.16535 | 0.183285 | 0.1586 | 0.1023 | 0.0163 | 0.0104 | 0.0102 |
| 10 | 0.586894 | 0.31299 | 0.22438 | 0.17129 | 0.1023 | 0.0132 | 0.1206 | 0.0142 | 0.0134 |
| 12 | 0.56540 | 0.3028 | 0.15675 | 0.16231 | 0.0163 | 0.1205 | 0.1093 | 0.0875 | 0.0056 |
| 14 | 0.54680 | 0.291299 | 0.14321 | 0.132456 | 0.0348 | 0.0419 | 0.1021 | 0.0342 | 0.00312 |
| 17 | 0.53020 | 0.27299 | 0.13222 | 0.11678 | 0.1251 | 0.0018 | 0.0952 | 0.0654 | 0.0351 |
| 19 | 0.53020 | 0.24098 | 0.11675 | 0.104596 | 0.011789 | 0.0011 | 0.07546 | 0.00239 | 0.0076 |

The C.V. Plot has been displayed in figure (5) as function of q by fixing $\alpha$ which verifies that it is decreasing function of q, i.e as q increases, C.V. decreases. In addition to C.V., we can also find Index of Dispersion(IOD)

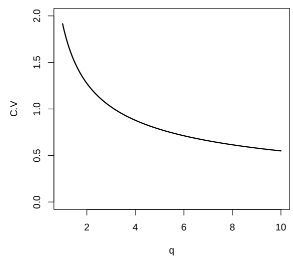


Figure 5: C.V. Plot

which is mathematically given by

$$IOD = \frac{Var(X)}{E(X)} = (\alpha+1)(q+1)\left(\frac{1}{(\alpha+2)(q+2)} - \frac{\alpha q}{(\alpha+\alpha q+q+1)^2}\right).$$

The numerical values of IOD for taking different values of $\alpha$ and $q$ are displayed in table (2).
Furthermore, The IOD plot has been exhibited in figure (6) with respect to q which shows that IOD is an increasing function of q.

We can also find Coefficient of Skewness ($\gamma_1$) and Measure of Kurtosis ($\gamma_2$) of the $\mathbb{ISP}(\alpha, q)$ distribution, respectively as:

$$\gamma_1 = \frac{\mu_3}{\mu_2^{\frac{3}{2}}}, \qquad \gamma_2 = \frac{\mu_4}{\mu_2^2}$$

The contour plot of both the Skewness and Kurtosis for parameters $\alpha$ and $q$ are shown in Figure 7.

Table 2: IOD values for various choices of parameters $\alpha$ and $q$

| q ↓ | $\alpha = 1$ | $\alpha = 2.5$ | $\alpha = 4$ | $\alpha = 6$ | $\alpha = 8$ | $\alpha = 10$ | $\alpha = 12$ | $\alpha = 14$ | $\alpha = 16$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.1944 | 0.1614 | 0.1556 | 0.1547 | 0.1435 | 0.1235 | 0.1087 | 0.09085 | 0.0768 |
| 2.5 | 0.161376 | 0.0947 | 0.0767 | 0.0683 | 0.0571 | 0.0431 | 0.0394 | 0.0231 | 0.0134 |
| 4 | 0.1556 | 0.0767 | 0.0544 | 0.0435 | 0.0342 | 0.0215 | 0.0162 | 0.0098 | 0.0056 |
| 6 | 0.1547 | 0.0683 | 0.0435 | 0.0309 | 0.0234 | 0.0178 | 0.0078 | 0.0543 | 0.0451 |
| 8 | 0.1556 | 0.0651 | 0.0389 | 0.0256 | 0.0134 | 0.0100 | 0.0098 | 0.0065 | 0.0035 |
| 10 | 0.1435 | 0.0636 | 0.0367 | 0.0228 | 0.01133 | 0.00986 | 0.8764 | 0.0061 | 0.0051 |
| 12 | 0.1345 | 0.0624 | 0.03567 | 0.0221 | 0.0127 | 0.0078 | 0.0123 | 0.00876 | 0.0067 |
| 14 | 0.1267 | 0.0611 | 0.03178 | 0.0211 | 0.0045 | 0.0023 | 0.2200 | 0.021235 | 0.01156 |
| 17 | 0.1134 | 0.0546 | 0.0298 | 0.0145 | 0.0134 | 0.01156 | 0.0234 | 0.00034 | 0.00027 |
| 19 | 0.1056 | 0.0467 | 0.0279 | 0.00934 | 0.0987 | 0.01234 | 0.01139 | 0.00012 | 0.00098 |

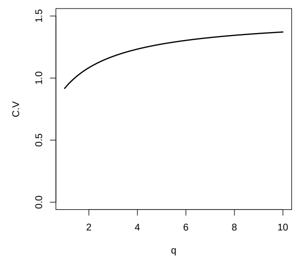


Figure 6: IOD Plot

*5.3. Geometric Mean*

The geometric mean ($G_Y$) of a rv $Y \sim \mathbb{ISP}(\alpha, q)$ is:

$$G_Y = E[\ln(Y)] = \int_0^1 \ln(y) f(y; \alpha, q) dy$$
$$= e^{-\frac{(\alpha+q)}{\alpha q}}.$$

*5.4. Harmonic Mean:*

The harmonic mean ($H_Y$) of a $\mathbb{ISP}(\alpha, q)$ distribution is:

$$H_Y = \frac{1}{E[\frac{1}{Y}]}$$
$$= \frac{(\alpha - 1)(q - 1)}{\alpha q}.$$

*5.5. L-Moments*

To explore the shape of a probability function, L-moments are used. The first L-moment is the mean of the density function. Explicit higher L-moment formula available. Nevertheless, the general formula for the $r^{th}$ L-moment ($r \geq 2$) is given by:

$$\lambda_k = \frac{1}{k} \sum_{i=0}^{k-2} (-1)^i \begin{pmatrix} k-2 \\ i \end{pmatrix} \begin{pmatrix} k \\ i+1 \end{pmatrix} i(-i + k - 1, i + 1), \tag{13}$$

where $J(i_1, i_2) = \int_0^1 F^{i_1}(y)(1 - F(y))^{i_2}\, dy$. In case of the $\mathbb{ISP}(\alpha, q)$ distribution

$$J(i_1, i_2) = \int_0^1 \left(\frac{qy^\alpha - \alpha y^q}{q - \alpha}\right)^{i_1} \left(1 - \frac{qy^\alpha - \alpha y^q}{q - \alpha}\right)^{i_2} dy.$$

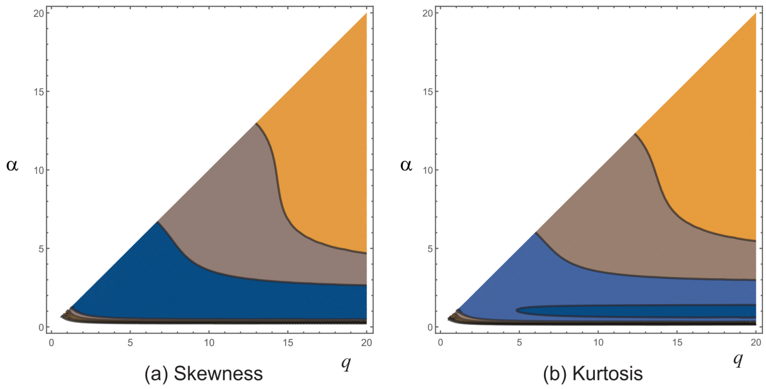


Figure 7: Plot of Skewness and Kurtosis

Here just the scale measure which is the second L-moment is given by:

$$\lambda 2 = \frac{\alpha q \left(2\alpha^2 + 3\alpha + 2\alpha q + q(2q+3) + 1\right)}{(\alpha+1)(2\alpha+1)(q+1)(2q+1)(\alpha+q+1)}, \quad \alpha > 0, q > 0. \tag{14}$$

*5.6. L-Skewness:*

The L-Skewness denoted by $\tau_3$ can be obtained as:

$$\tau_3 = \frac{\lambda_3}{\lambda_2}.$$

*5.7. L-Kurtosis:*

The L-Kurtosis ($\tau_4$) is defined as:

$$\tau_4 = \frac{\lambda_4}{\lambda_2}.$$

Contour plot of both L-skewness and L-kurtosis are shown in Figure 8.

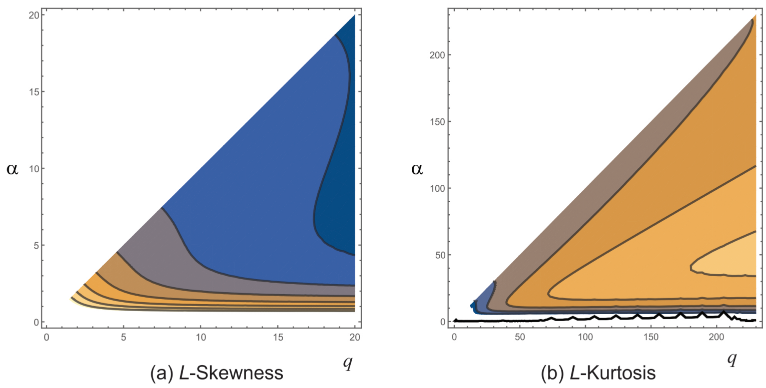


Figure 8: Plot of L-skewness and L-kurtosis

## 6. Methods of Estimation

*6.1. Method of Moments*

Moments estimator of parameter $\alpha$ and $q$ of $\mathbb{ISP}(\alpha, q)$ can be find out easily by solving

$$m_1 = \mu_1' = \frac{\alpha q}{(\alpha+1)(q+1)}, \quad \text{and} \quad m_2 = \mu_2' = \frac{\alpha q}{(\alpha+2)(q+2)},$$

where $m_1$ and $m_2$ are first and second sample moments. Solving above system of equations in terms of $\alpha$ and $q$, the moment estimators obtained are

$$\hat{\alpha} = \frac{-3m_1m_2 - \sqrt{(3m_1m_2 + m_1 - 4m_2)^2 - 8m_1m_2(m_1m_2 + m_1 - 2m_2)} - m_1 + 4m_2}{2(m_1m_2 + m_1 - 2m_2)}, \tag{15}$$

$$\hat{q} = \frac{\sqrt{m_1^2(m_2-1)^2 - 8m_1m_2(m_2+1) + 16m_2^2} - m_1(3m_2+1) + 4m_2}{2(m_1m_2 + m_1 - 2m_2)}. \tag{16}$$

*6.2. Maximum Likelihood Estimation*

The popular method of obtaining estimates is MLE. Consider a random sample of size $n$ from the $\mathbb{ISP}(\alpha, q)$ distribution with pdf (2). The corresponding likelihood function is

$$L(\alpha, q|\underline{\mathbf{y}}) = \left(\frac{\alpha q}{q-\alpha}\right)^n \sum_{i=0}^{n} (y^{\alpha-1} - y^{q-1}). \tag{17}$$

Taking log of (17), we get:

$$\log(L(\alpha, q|\underline{\mathbf{y}})) = n\log(\alpha) + n\log q - n\log(q-\alpha) + \sum_{i=0}^{n} \log(y_i^{\alpha-1} - y_i^{q-1}). \tag{18}$$

The ML Estimates $\hat{\alpha}$ of $\alpha$ and $\hat{q}$ of $q$, respectively, can be obtained by solving equations

$$\frac{\partial \log L}{\partial \alpha} = 0, \quad \text{and} \quad \frac{\partial \log L}{\partial q} = 0.$$

where

$$\frac{\partial \log L}{\partial \alpha} = \frac{n}{\alpha} + \frac{n}{q-\alpha} + \sum_{i=0}^{n} \frac{y^{\alpha-1}\log(y)}{y^{\alpha-1} - y^{q-1}},$$

and

$$\frac{\partial \log L}{\partial q} = \frac{n}{q} - \frac{n}{q-\alpha} - \sum_{i=0}^{n} \frac{y^{q-1}\log(y)}{y^{\alpha-1} - y^{q-1}}.$$

Unfortunately, above equations are not in explicit forms and therefore a suitable iterative procedure is needed to get the required estimates numerically.
The second order partial derivatives of (18) are as follows:

$$\begin{aligned}
\frac{\partial^2 \log L}{\partial \alpha^2} &= -\frac{n}{\alpha^2} + \frac{n}{(q-\alpha)^2} + \sum_{i=0}^{n} \left( \frac{y^{\alpha-1}\log^2(y)}{y^{\alpha-1} - y^{q-1}} - \frac{y^{2\alpha-2}\log^2(y)}{(y^{\alpha-1} - y^{q-1})^2} \right), \\
\frac{\partial^2 \log L}{\partial \alpha \partial q} &= \sum_{i=0}^{n} \frac{\log^2(y) y^{\alpha+q}}{(y^q - y^\alpha)^2} - \frac{n}{(q-\alpha)^2}, \\
\frac{\partial^2 \log L}{\partial q^2} &= -\frac{n}{q^2} + \frac{n}{(q-\alpha)^2} - \sum_{i=0}^{n} \left( \frac{y^{q-1}\log^2(y)}{y^{\alpha-1} - y^{q-1}} + \frac{y^{2q-2}\log^2(y)}{(y^{\alpha-1} - y^{q-1})^2} \right).
\end{aligned}$$

Obtaing the expected Fisher information matrix as

$$\mathbf{J_x} = \begin{bmatrix} -\mathbb{E}\left(\frac{\partial^2 \log L}{\partial \alpha^2}\right) & -\mathbb{E}\left(\frac{\partial^2 \log L}{\partial \alpha \partial q}\right) \\ -\mathbb{E}\left(\frac{\partial^2 \log L}{\partial q \partial \alpha}\right) & -\mathbb{E}\left(\frac{\partial^2 \log L}{\partial q^2}\right) \end{bmatrix}$$

which in approximation can be written ass

$$\mathbf{J_x} \approx \begin{bmatrix} J_{\alpha\alpha} & J_{\alpha q} \\ J_{q\alpha} & J_{qq} \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 \log L}{\partial \alpha^2}\Big|_{\hat{\alpha},\hat{q}} & \frac{\partial^2 \log L}{\partial \alpha \partial q}\Big|_{\hat{\alpha},\hat{q}} \\ \\ \frac{\partial^2 \log L}{\partial q \partial \alpha}\Big|_{\hat{\alpha},\hat{q}} & \frac{\partial^2 \log L}{\partial q^2}\Big|_{\hat{\alpha},\hat{q}} \end{bmatrix}$$

where $\hat{\alpha}$ and $\hat{q}$ are MLE of $\alpha$ and $q$ respectively. Hence, when $n$ is large and under some mild regularity conditions,

$$\sqrt{n}\begin{pmatrix} \alpha - \hat{\alpha} \\ q - \hat{q} \end{pmatrix} \overset{a}{\sim} N_2\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{J_x}^{-1}\right),$$

where " $\overset{a}{\sim}$ " means approximately distributed, and $\mathbf{J_x}^{-1}$ is the inverse of $\mathbf{J_x}$. The above asymptotic normal distribution is needful for the construction of approximate confidence intervals for the parameters.

**Generation of Random Numbers**

Usually for generating random numbers of any arbitrary distribution, the Inverse CDF technique is used. However, sometimes due to the implicit form of distribution function of the proposed model, it becomes cumbersome to generate random numbers by using this technique. In our proposed model we discuss an alternative way to generate the random variables for $\mathbb{ISP}(\alpha, q)$. As the proposed model is derived by taking the inverse of Slashed Pareto distribution, following algorithm can be used to get random numbers of $\mathbb{ISP}(\alpha, q)$:

**Step 1:** Generate $U_i$ and $V_i$ $(i = 1, 2, \ldots, n)$ from $U(0, 1)$ independently.

**Step 2:** Corresponding to each $V_i$, determine $X_i = (1 - V_i)^{\frac{-1}{\alpha}}$.

**Step 3:** Finally generate $Y_i$ from $\frac{U_i^{\frac{1}{q}}}{x_i}$.

## 7. Simulation Study

In this section, we perform simulation of the proposed model to evaluate the performance of ML estimators $\hat{\alpha}$ and $\hat{q}$ in estimating $\alpha$ and $q$, respectively. Simulation was accomplished with the help of `R` compuational software (`R code` of simulation study can be available on request for the reader), and the number of replications was 10000. The assessment of each point estimate was carried out on the basis of the average bias and the mean squared error (MSE) for each sample size, whose respective formulas are given as under:

$$\frac{1}{m}\sum_{i=1}^{m}\left(\hat{\Lambda}_i - \Lambda_0\right),$$

The average MSE

$$\frac{1}{m}\sum_{i=1}^{m}\left(\hat{\Lambda}_i - \Lambda_0\right)^2.$$

We took the sample size of $n = 100, 200, 300, 400$ and consider $\alpha = 0.5$, $1.5$, $2.5$ and $3.5$ and $q = 4, 6$. The simulation results for each parameter $\alpha$ and $q$ are displayed in Tables 3 and 4, respectively.

Table 3: Simulation study for $\alpha$ estimates based on MLE

| | q=4 | | | | | q=6 | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | $\alpha$ | | | | | $\alpha$ | | |
| $\alpha$=0.5 | n | Mean | Bias | MSE | Var | n | Mean | Bias | MSE | Var |
| | 100 | 0.52012 | 0.02012 | 6.00E-05 | 0.00606 | 100 | 0.51495 | 0.01495 | 5.00E-05 | 0.0044 |
| | 200 | 0.50825 | 0.00825 | 1.00E-05 | 0.00212 | 200 | 0.50684 | 0.00684 | 1.00E-05 | 0.00174 |
| | 300 | 0.50527 | 0.00527 | 0 | 0.00131 | 300 | 0.50513 | 0.00513 | 0 | 0.00109 |
| | 400 | 0.5039 | 0.0039 | 0 | 0.00095 | 400 | 0.50354 | 0.00354 | 0 | 8.00E-04 |
| $\alpha$=1.5 | n | Mean | Bias | MSE | Var | n | Mean | Bias | MSE | Var |
| | 100 | 1.65197 | 0.15197 | 0.00151 | 0.12841 | 100 | 1.60789 | 0.10789 | 0.00115 | 0.10384 |
| | 200 | 1.59187 | 0.09187 | 4.00E-04 | 0.07235 | 200 | 1.54689 | 0.04689 | 0.00021 | 0.04043 |
| | 300 | 1.56178 | 0.06178 | 0.00017 | 0.0477 | 300 | 1.52799 | 0.02799 | 8.00E-05 | 0.02179 |
| | 400 | 1.54817 | 0.04817 | 9.00E-05 | 0.03447 | 400 | 1.52102 | 0.02102 | 4.00E-05 | 0.01446 |
| $\alpha$=2.5 | n | Mean | Bias | MSE | Var | n | Mean | Bias | MSE | Var |
| | 100 | 2.6363 | 0.1363 | 0.0027 | 0.25129 | 100 | 2.7518 | 0.2518 | 0.00418 | 0.35425 |
| | 200 | 2.63502 | 0.13502 | 0.00105 | 0.19236 | 200 | 2.66595 | 0.16595 | 0.00123 | 0.21924 |
| | 300 | 2.62413 | 0.12413 | 0.00059 | 0.16105 | 300 | 2.62597 | 0.12597 | 0.00058 | 0.15798 |
| | 400 | 2.62066 | 0.12066 | 0.00038 | 0.13906 | 400 | 2.59389 | 0.09389 | 0.00031 | 0.11604 |
| $\alpha$=3.5 | n | Mean | Bias | MSE | Var | n | Mean | Bias | MSE | Var |
| | 100 | 3.33589 | 0.16411 | 0.00362 | 0.33503 | 100 | 3.74219 | 0.24219 | 0.00614 | 0.55502 |
| | 200 | 3.35552 | 0.14448 | 0.00132 | 0.24261 | 200 | 3.72964 | 0.22964 | 0.00224 | 0.39526 |
| | 300 | 3.37479 | 0.12521 | 0.00072 | 0.19942 | 300 | 3.70621 | 0.20621 | 0.00124 | 0.32873 |
| | 400 | 3.38556 | 0.11444 | 0.00048 | 0.17825 | 400 | 3.68676 | 0.18676 | 0.00082 | 0.29159 |

Based on the results from Simulation study, we can claim that:

- As expected, MSE and Bias for all estimators decreases as sample size increases which confirms the attainment of stability of estimators.

## 8. Numerical Illustration

In this section, the applicability of $\mathbb{ISP}(\alpha, q)$ has been shown by considering two data sets corresponding to the Households with Access to Safe Drinking Water of the 35 states in 2011 in India. They were extracted from the Households with Access to safe Drinking Water The proposed distribution has been compared with following distributions namely:

(i). Beta Distribution (BD):

$$f_1(y) = \frac{1}{B(\alpha,\beta)} y^{\alpha-1}(1-y)^{\beta-1}, \quad \alpha, \beta > 0.$$

(ii). Kumarswamy's Distribution (KSD):

$$f_2(y) = \alpha\beta y^{\alpha-1}\left(1-y^{\alpha}\right)^{\beta-1}, \quad \alpha, \beta > 0.$$

Table 4: Simulation study for $q$ estimates based on MLE

| q=4 | | | | | q=6 | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | q | | | | | q | | |
| n | mean | Bias | MSE | Var | n | mean | Bias | MSE | Var |
| 100 | 5.573071 | 1.573071 | 1.417144 | 139.2537 | 100 | 9.60985 | 3.60985 | 4.360972 | 423.1085 |
| 200 | 4.408995 | 0.408995 | 0.062122 | 12.25835 | 200 | 7.186886 | 1.186886 | 0.467649 | 92.13032 |
| 300 | 4.216082 | 0.216082 | 0.008543 | 2.516459 | 300 | 6.502878 | 0.502878 | 0.031402 | 9.168647 |
| 400 | 4.1522 | 0.1522 | 0.003519 | 1.38453 | 400 | 6.393742 | 0.393742 | 0.016025 | 6.255635 |
| | | | | | | | | | |
| n | mean | Bias | MSE | Var | n | mean | Bias | MSE | Var |
| 100 | 4.289087 | 0.289087 | 0.170197 | 16.93786 | 100 | 6.979715 | 0.979715 | 1.589365 | 157.9924 |
| 200 | 4.051729 | 0.051729 | 0.008702 | 1.737991 | 200 | 6.260078 | 0.260078 | 0.023757 | 4.684144 |
| 300 | 4.050005 | 0.050005 | 0.003806 | 1.139303 | 300 | 6.146147 | 0.146147 | 0.008682 | 2.583469 |
| 400 | 4.014399 | 0.014399 | 0.002019 | 0.807526 | 400 | 6.105994 | 0.105994 | 0.004652 | 1.849853 |
| | | | | | | | | | |
| n | mean | Bias | MSE | Var | n | mean | Bias | MSE | Var |
| 100 | 4.464601 | 0.464601 | 0.126949 | 12.48026 | 100 | 6.469776 | 0.469776 | 0.250139 | 24.79569 |
| 200 | 4.148039 | 0.148039 | 0.007721 | 1.522517 | 200 | 6.088095 | 0.088095 | 0.019621 | 3.916824 |
| 300 | 4.085444 | 0.085444 | 0.003709 | 1.105363 | 300 | 6.020559 | 0.020559 | 0.008439 | 2.531534 |
| 400 | 4.020378 | 0.020378 | 0.002063 | 0.824854 | 400 | 6.014956 | 0.014956 | 0.004625 | 1.84985 |
| | | | | | | | | | |
| n | mean | Bias | MSE | Var | n | mean | Bias | MSE | Var |
| 100 | 4.923461 | 0.923461 | 0.095216 | 8.669718 | 100 | 6.59762 | 0.59762 | 0.143692 | 14.01349 |
| 200 | 4.586259 | 0.586259 | 0.009753 | 1.607132 | 200 | 6.122012 | 0.122012 | 0.017142 | 3.413757 |
| 300 | 4.450452 | 0.450452 | 0.004033 | 1.007121 | 300 | 6.022998 | 0.022998 | 0.007958 | 2.387207 |
| 400 | 4.398152 | 0.398152 | 0.002419 | 0.809229 | 400 | 5.989169 | 0.010831 | 0.004818 | 1.927462 |

(iii). Log-Lindley Distribution (LLD):

$$f_3(y) = \sigma(\lambda + \sigma(\lambda - 1)\log(y))y^{\sigma-1}, \quad \sigma > 0, 0 \leq \lambda \leq 1.$$

Proposed model including the competing ones have been compared by using log-likelihood (LL), Akaike's Information Criterion (AIC)[1] and Bayesian information criterion (BIC) [10]. To check the goodness of fit, empirical distribution function (EDF) goodness-of-fit measures like KolmogorovSmirnov (KS) test statistics, the Cramervon Mises (CVM) test statistics, and the AndersonDarling (AD) test statistics have been used and whose definition and formulas are given as under:
Denote the cdf of the fitted model by $\hat{F}$, the original data by $y_1, ..., y_M$, and the ordered data in increasing magnitude by $y_{(1)}, \ldots, y_{(M)}$, then we have

1. KS test statistics: D = $\max(D^+, D^-)$, where

$$D^+ = \max_{1\leq k\leq M} \mid \frac{k}{M} - \hat{F}(y_{(k)}) \mid,$$

$$D^- = \max_{1\leq k\leq M} \mid \hat{F}(y_{(k)}) - \frac{k-1}{M} \mid.$$

2. CVM test statistics:

$$W^2 = \sum_{k=1}^{M} \left[ \hat{F}(y_{(k)}) - \frac{2k-1}{2M} \right]^2 + \frac{1}{12M}.$$

3. AD test statistics:

$$A^2 = -M - \frac{1}{M} \sum_{k=1}^{M} \left[ (2k-1)\log(\hat{F}(y_{(k)})) + (2m+1-2k)\log(1-\hat{F}(y_{(k)})) \right].$$

Table 5: Model validation criterion of different probabilistic models for dataset-1

| Models | Estimated Prameters | LL | AIC | BIC |
|---|---|---|---|---|
| ISP | $\alpha$=3.67362, $q$ = 270.367 | 19.9994 | 39.9988 | 36.8881 |
| BD | $\alpha$=3.35461, $\beta$=0.914946 | 19.8176 | 39.6352 | 36.5245 |
| KSD | $\alpha$= 3.43322, $\beta$= 0.922136 | 19.7934 | 39.5868 | 36.4761 |
| LLD | $\lambda$= 27308.5, $\sigma$ =3.62441 | 19.7257 | 39.4514 | 36.3407 |

Table 6: Model validation criterion of different probabilistic models for dataset-2

| Models | Estimated Prameters | LL | AIC | BIC |
|---|---|---|---|---|
| ISP | $\alpha$= 3.04388, $q$ =815.987 | 15.462 | 30.924 | 27.8133 |
| BD | $\alpha$= 2.93138, $\beta$ =0.960894 | 15.3887 | 30.7774 | 27.6667 |
| KSD | $\alpha$= 2.95573, $\beta$= 0.96144 | 15.3867 | 30.7734 | 27.6627 |
| LLD | $\lambda$=35280.6, $\sigma$ =3.03259 | 15.3707 | 30.7414 | 27.6307 |

Table 7: EDF goodness-of-fit measures of different distributions for dataset-1

| Test | ISP | BD | KSD | LLD |
|---|---|---|---|---|
| KS | 0.136 (0.575) | 0.148(0.541) | 0.146 (0.361) | 0.137(0.443) |
| CVM | 0.124(0.58) | 0.150(0.325) | 0.130(0.471) | 0.132(0.455) |
| AD | 0.643(0.737) | 0.659 (0.466) | 0.678 (0.551) | 0.785 (0.638) |

The parameter estimates of both the data sets taken into consideration for each model along with LL, AIC and BIC are computed and tabulated in tables 5 and 6, respectively and the goodness of fit for each data set is presented in table 7 and 8. From all these tables we can claim that the superiority of the proposed model is established. Furthermore, Figure 9 and 10 shows the PP plot for both data set 1 and data set 2 respectively.

Table 8: EDF goodness-of-fit measures of different distributions for dataset-2

| Test | ISP | BD | KSD | LLD |
|---|---|---|---|---|
| KS | 0.096(0.644) | 0.099(0.597) | 0.103(0.497) | 0.112(0.509) |
| CVM | 0.046(0.707) | 0.052(0.612) | 0.059(0.685) | 0.058(0.696) |
| AD | 0.303(0.874) | 0.465(0.756) | 0.368(0.665) | 0.394(0.705) |

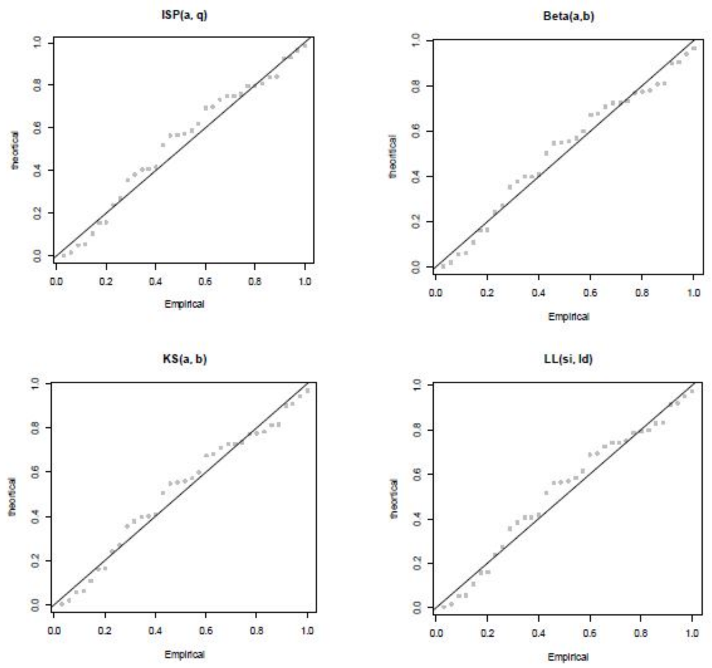


Figure 9: PP Plot for Data Set 1

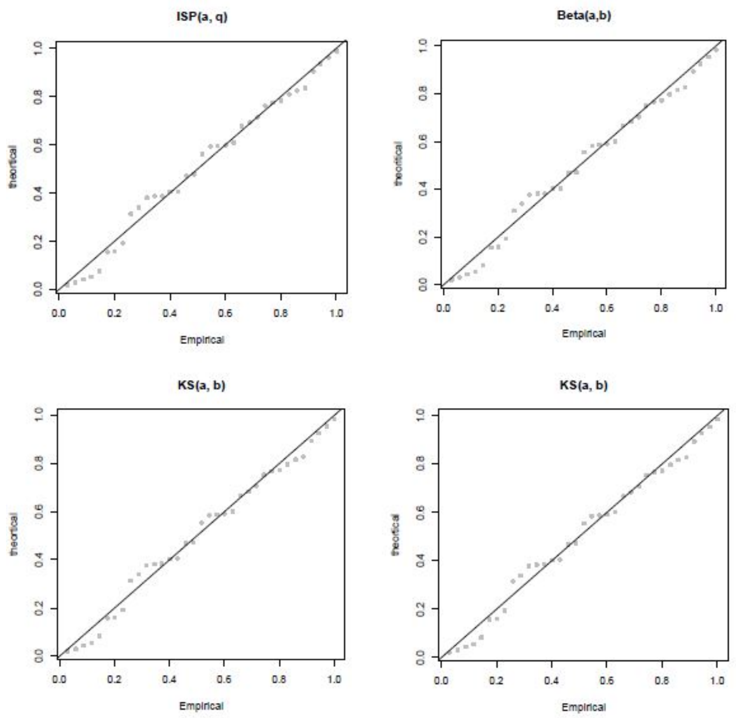


Figure 10: PP Plot for Data Set 2

## 9. Conclusion

Here in this work, we introduced a new two parameter continuous model with bounded support (0,1). This new model, the Inverse-Slashed Pareto distribution, has been accomplished by simply taking the inverse of Slashed Pareto distribution. This new model being very simple and have some satisfying properties. From application point of view, two data sets have been considered to explore its superiority over its competing models. We are hopeful that our new distribution will be highly useful and can have significant contributions across all the relevant fields of statistical sciences.

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