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Comments on the Suzuki type fixed point theorems

In memories of Art Kirk, Kaz Goebel, and Wataru Takahashi

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Abstract

There have appeared thousands of works about the Banach contraction and its extensions on metric spaces (X, d). Recently, many works on the Rus-Hicks-Rhoades (RHR) maps $T : X \to X$ satisfying $d(Tx, T^2x) \leq \alpha\, d(x, Tx)$ on $x \in X$ with $\alpha \in [0, 1)$ also appeared. In the present article, we show that the so-called Suzuki type maps are RHR maps and the proofs of results of Suzuki and his colleagues can be simplified within a few lines based on our recent works on quasi-metric spaces.

Keywords: Banach contraction, Rus-Hicks-Rhoades map, Suzuki type maps, fixed point, quasi-metric, maximal element.
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1. Introduction

Let (X, d) be a metric space. A Banach contraction $T : X \to X$ is a map satisfying

$$d(Tx, Ty) \leq \alpha\, d(x, y) \quad \text{for all } x, y \in X$$

with some $\alpha \in [0, 1)$. There have been appeared thousands of articles related to the Banach contraction.

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Recently, we introduced the Rus-Hicks-Rhoades (RHR) map $T : X \to X$ satisfying

$$d(Tx, T^2x) \leq \alpha\, d(x, Tx) \quad \text{for all } x \in X$$

with some $\alpha \in [0, 1)$. See our recent works [31]–[35].

The RHR maps are called graphic contraction, iterative contraction, weakly contraction, or Banach mapping; see Berinde et al. [4],[5]. Moreover, it is recently known that well-known metric fixed point theorems related to the RHR maps hold for quasi-metric spaces (without assuming the symmetry); see [32]–[35].

Recall that Suzuki [45] in 2008 gave a very interesting result on particular RHR maps. It is a weaker version of the Banach contraction principle and also characterizes the completeness of underlying metric spaces. It was commonly regarded that Suzuki's result gave a new direction to the subject and as a result, researchers including his colleagues made many contributions in metric fixed point theory. Most of such results are variations and refinements of Suzuki's original result with lengthy complicated proofs. However, we found that most of such Suzuki type results are incorrectly stated according to our recently developed ordered fixed point theory.

Our aim in this article is to collect results of Suzuki and his colleagues related to Suzuki type maps. We give few line proofs of them based on our ordered fixed point theory [30],[33]-[35].

This article is organized as follows: Section 2 is preliminaries on quasi-metric spaces. In Section 3, we give basic theorems on the Rus-Hicks-Rhoades (RHR) maps and extensions of the Banach contraction principle for a quasi-metric space (X, δ) with a selfmap $T : X \to X$ such that X is T-orbitally complete. In Section 4, we collect whole RHR type theorems due to Suzuki and his colleagues with few line proofs if necessary. Section 5 deals with the contents of our previous works related to RHR maps. In such works, we suggested that Suzuki's theorems can be easily proved, but no one responded yet. Finally, in Section 6, we give some conclusion.

2. Preliminaries

We recall the following:

Definition 2.1. A *quasi-metric* on a nonempty set X is a function $\delta : X \times X \to [0, \infty)$ satisfying the following conditions for all $x, y, z \in X$:

(a) (self-distance) $\delta(x, y) = \delta(y, x) = 0 \iff x = y$;
(b) (triangle inequality) $\delta(x, z) \leq \delta(x, y) + \delta(y, z)$.

A *metric* on a set X is a quasi-metric satisfying

(c) (symmetry) $\delta(x, y) = \delta(y, x)$ for all $x, y \in X$.

For quasi-metric spaces, the convergence of a sequence, Cauchy sequences, completeness, orbits, and orbital continuity are routinely defined as follows:

Definition 2.2. ([1], [16])

(1) A sequence (x_n) in X *converges* to $x \in X$ if

$$\lim_{n\to\infty} \delta(x_n, x) = \lim_{n\to\infty} \delta(x, x_n) = 0.$$

(2) A sequence (x_n) is *left-Cauchy* if for every $\varepsilon > 0$, there is a positive integer $N = N(\varepsilon)$ such that $\delta(x_n, x_m) < \varepsilon$ for all $n > m > N$.

(3) A sequence (x_n) is *right-Cauchy* if for every $\varepsilon > 0$, there is a positive integer $N = N(\varepsilon)$ such that $\delta(x_n, x_m) < \varepsilon$ for all $m > n > N$.

(4) A sequence (x_n) is *Cauchy* if for every $\varepsilon > 0$ there is positive integer $N = N(\varepsilon)$ such that $\delta(x_n, x_m) < \varepsilon$ for all $m, n > N$; that is (x_n) is a *Cauchy sequence* if it is left and right Cauchy.

Definition 2.3. ([1], [16])

(1) (X, δ) is *left-complete* if every left-Cauchy sequence in X is convergent;
(2) (X, δ) is *right-complete* if every right-Cauchy sequence in X is convergent;
(3) (X, δ) is *complete* if every Cauchy sequence in X is convergent.

Definition 2.4. Let (X, δ) be a quasi-metric space and $T : X \to X$ a selfmap. The *orbit* of T at $x \in X$ is the set

$$O_T(x) = \{x, T(x), \cdots, T^n(x), \cdots\}.$$

The space X is said to be *T-orbitally complete* if every right-Cauchy sequence in $O_T(x)$ is convergent in X. A selfmap T of X is said to be *orbitally continuous* at $x_0 \in X$ if

$$\lim_{n\to\infty} T^n(x) = x_0 \Longrightarrow \lim_{n\to\infty} T^{n+1}(x) = T(x_0)$$

for any $x \in X$.

Note that every complete metric space is T-orbitally complete for all maps $T : X \to X$. There exists a T-orbitally complete metric space but it is not complete. Moreover, there exists an orbitally continuous map but it is not continuous.

Every quasi-metric induces a metric, that is, if (X, δ) is a quasi-metric space, then the function $d : X \times X \to [0, \infty)$ defined by

$$d(x, y) = \max\{\delta(x, y), \delta(y, x)\}$$

is a metric on X; see Jleli *et al.* [16].

The following was given in [34]:

Theorem 2.5. *A selfmap $T : X \to X$ of a quasi-metric space (X, δ) has a fixed point $z \in X$ if and only if z is a fixed point of the selfmap T of the induced metric space (X, d).*

Proof. If $z = T(z)$ in (X, δ), then

$$d(z, T(z)) = \max\{\delta(z, T(z)), \delta(T(z), z)\} = 0,$$

and hence $d(z, T(z)) = 0$. The converse is true for $d = \delta$. $\square$

From this, all metric fixed point theorems are true for quasi-metric spaces. This is a rather surprising fact in the one hundred year history of the metric fixed point theory since Banach in 1922.

3. Basic Theorems

In this section, we give extensions of the Banach contraction principle called the Rus-Hicks-Rhoades theorem for a quasi-metric space (X, δ) with a selfmap $T : X \to X$ such that X is T-orbitally complete.

The following form of the RHR theorem is a consequence of Theorems H and P in [30], [33], [34] and useful in this article:

Theorem H(γ1). *Let (X, δ) be a quasi-metric space, and $0 < \alpha < 1$. If $f : X \to X$ is a map satisfying*

$$\delta(f(x), f^2(x)) \le \alpha\, \delta(x, f(x)) \text{ for all } x \in X \backslash \{f(x)\},$$

and X is f-orbitally complete, then f has a fixed point $v \in X$, that is, $v = f(v)$.

Recall that the Banach contraction principle was extended to multimaps by Nadler [24] in 1969. A more general form of Nadler's theorem was established by Covitz-Nadler [7] in 1970 for metric spaces.

Let (X, δ) be a quasi-metric space and $\mathrm{Cl}(X)$ denote the family of all nonempty closed subsets of X (not necessarily bounded). For $A,\ B \in \mathrm{Cl}(X)$, set

$$H(A,B) = \begin{cases} \max\{\sup\{\delta(a,B) : a \in A\},\ \sup\{\delta(b,A) : b \in B\}\} & \text{if the maximum exists} \\ \infty & \text{if otherwise.} \end{cases} \tag{1}$$

where $\delta(a,B) = \inf\{\delta(a,b) : b \in B\}$.

Such a map H is called generalized Hausdorff quasi-metric induced by δ. Notice that H is a quasi-metric on $\mathrm{Cl}(X)$. A point $p \in X$ is called a fixed point of $T : X \to \mathrm{Cl}(X)$ if $p \in T(p)$. A function $f : X \to \mathbb{R}$ is said to be T-orbitally lower semi-continuous if $\{x_n\}$ is a sequence in $O(T, x_0)$ and $x_n \to \zeta$ implies $f(\zeta) \le \lim_n \inf f(x_n)$.

Moreover, we have the following from Theorem H in [30],[33],[34]:

Theorem H$(\delta 1)$**.** *Let (X, δ) be a complete quasi-metric space, and $0 < \alpha < 1$. Let $T : X \to \mathrm{Cl}(X)$ be a multimap such that, for any $x \in X \backslash T(x)$, there exists $y \in X \backslash \{x\}$ satisfying*

$$H(T(x), T(y)) \le \alpha\, \delta(x,y).$$

Then T has a fixed point $v \in X$, that is, $v \in T(v)$.

The following is given in [34]:

Theorem P. *Let (X, δ) be a quasi-metric space and let $T : X \to X$ be an RHR map; that is,*

$$\delta(T(x), T^2(x)) \le \alpha\, \delta(x, T(x)) \text{ for every } x \in X,$$

where $0 < \alpha < 1$.

(i) *If X is T-orbitally complete, then, for each $x \in X$, there exists a point $x_0 \in X$ such that*

$$\lim_{n\to\infty} T^n(x) = x_0$$

and

$$\delta(T^n(x), x_0) \le \frac{\alpha^n}{1-\alpha}\delta(x, T(x)), \quad n = 1, 2, \cdots .$$

(ii) *x_0 is a fixed point of T, and, equivalently,*

(iii) *$T : X \to X$ is orbitally continuous at $x_0 \in X$.*

Theorem P was originated from Hicks-Rhoades [12]. The first example of Theorem P was given by Kannan in 1969.

We are going to show that most results on RHR maps due to Suzuki and his colleagues are simple consequences of the three theorems in this section.

4. Theorems due to Suzuki et al.

From 2008, Suzuki found fixed point theorems for certain RHR type maps with very sophisticated proofs. He and his colleagues continued to publish more papers within a few years. Their papers became very popular and many followers published scores of papers of the same nature.

In this section, we collect whole RHR type theorems due to Suzuki and his colleagues with few line proofs if necessary.

Suzuki [43] in 2001

For metric spaces, there have appeared thousands of generalizations, modifications, imitations, extensions, and their applications to fixed point theorems. One of them for RHR maps is the following:

Theorem 1. (Suzuki) *Let X be a complete metric space and let T be a mapping from X into itself. Suppose that there exist a τ-distance p on X and $r \in [0,1)$ such that $p(Tx, T^2x) \leq r \cdot p(x, Tx)$ for all $x \in X$. Assume that either of the following holds:*

(i) *If $\lim_n \sup\{p(x_n, x_m) : m > n\} = 0$, and $\lim_n p(x_n, Tx_n) = 0$, and $\lim_n p(x_n, y) = 0$, then $Ty = y$;*

(ii) *If $\{x_n\}$ and $\{Tx_n\}$ converge to y, then $Ty = y$;*

(iii) *T is continuous.*

Then there exists $x_0 \in X$ such that $Tx_0 = x_0$ and $p(x_0, x_0) = x_0$.

For quasi-metric spaces, either of (i) or (ii) means $\{x_n\}$ is Cauchy and T is orbitally continuous. Theorem due to Rus [39] in 1973 is a particular case of (iii) and we showed several times that the continuity is redundant. It is open that whether this also holds for Theorem 1. Moreover, if p is a quasi-metric, Theorem 1 follows from our Theorem H(γ1).

There may be similar theorems for thousands of artificial metric type spaces.

Suzuki [44] in 2005

In this paper, Suzuki extends the generalized Caristi's fixed point theorems due to Bae [2] and Bae et al. [3].

From the original Caristi's theorem, Suzuki deduced the following in [44]:

Theorem 2. (Suzuki) *Let X be a complete metric space with metric d. Let T be a mapping from X into itself and let f be a lower semicontinuous function from X into $[0,\infty)$. Let φ be a function from X into $[0,\infty)$ satisfying*

$$\sup\{\varphi(x) : x \in X,\ f(x) \leq \inf_{w \in X} f(w) + \eta\} < \infty$$

for some $\eta > 0$. Assume that

$$d(x, Tx) \leq \varphi(x)(f(x) - f(Tx))$$

for all $x \in X$. Then there exists a fixed point $x_0 \in X$ of T.

Since the lower semicontinuity in the Caristi theorem was extended to the one *from above*, Suzuki's theorem and all of its consequences due to Bae et al. can also extended. Moreover, they may be extended to quasi-metric spaces. Consequently, all eleven theorems in Suzuki [44] can be improved.

Suzuki [45] in 2008

Suzuki generalized the Banach contraction principle as follows:

Theorem 2. (Suzuki) *Let (X, d) be a complete metric space and T be a mapping on X. Define a nonincreasing function θ from $[0,1)$ onto $(1/2, 1]$ by*

$$\theta(r) = \begin{cases} 1 & \text{if } 0 \leq r \leq (\sqrt{5}-1)/2, \\ (1-r)r^{-2} & \text{if } (\sqrt{5}-1)/2 \leq r \leq 2^{-1/2}, \\ (1+r)^{-1} & \text{if } 2^{-1/2} \leq r < 1. \end{cases} \tag{2}$$

Assume there exists $r \in [0,1)$ such that

$$\theta(r)d(x, Tx) \leq d(x, y) \text{ implies } d(Tx, Ty) \leq rd(x, y)$$

for all $x, y \in X$. Then there exists a unique fixed point z of T. Moreover $\lim_n T^n x = z$ for all $x \in X$.

New Proof. Note that T is an RHR map and Theorems P or H(γ1) can be applied. Now it suffices to show the uniqueness of the fixed point z. If $w \in X$ is a fixed point, then

$$\theta(r)d(z, Tz) \leq d(z, w) \text{ implies } d(Tz, Tw) \leq r\, d(z, w).$$

Hence $d(z, w) = d(Tz, Tw) \leq r\, d(z, w)$ and consequently $d(z, w) = 0$. □

There have been appeared a large number of variants of Suzuki's theorem. Many of them can be improved by easy proofs as shown as above.

From now on, T is called the Suzuki type as its many followers used it.

Suzuki noted that the following theorem says that $\theta(r)$ is the best constant for every $r \in [0, 1)$:

Theorem 3. *Define a function θ as in Theorem 2. Then for each $r \in [0, 1)$, there exist a complete metric space (X, d) and a mapping T on X such that T does not have a fixed point and*

$$\theta(r)d(x, Tx) < d(x, y) \textit{ implies } d(Tx, Ty) \leq r\, d(x, y)$$

for all $x, y \in X$.

New Proof. Note that $\theta(r)d(x, Tx) < d(x, y)$ means T can not have a fixed point $x = y = Tx$. □

In order to show uselessness of Theorems 2 and 3 of Suzuki, let us consider any function $\theta' : [0, \infty) \to [0, 1]$:

Theorem 2.′ *Replace the function θ in Theorem 2 by θ'. Then the conclusion of Theorem 2 holds.*

Proof. Note that, by putting $y = Tx$, T becomes an RHR map. Then by Theorems P or H(γ1), T has a fixed point and its uniqueness follows as in the proof of Theorem 2. □

Theorem 3.′ *Replace the function θ in Theorem 3 by θ'. Then the conclusion of Theorem 3 holds.*

Proof. Note that $\theta'(r)d(x, Tx) < d(x, y)$ means T can not have a fixed point $x = y = Tx$. □

Recall that there are hundreds of equivalent conditions for metric completeness, e.g. Kirk [22], Park [27], Cobzaş [6] as typical examples. Comments for them are given by Park and Rhoades [37] in 1986.

Kikkawa and Suzuki [18] in 2008

The authors prove three fixed point theorems for generalized contractions with constants in complete metric spaces, which are generalizations of fixed point theorems due to Suzuki in the same year.

Kikkawa-Suzuki [18] stated the following is in Suzuki [45] in 2008:

Theorem 1. (Suzuki) *For a metric space (X, d), the following are equivalent:*

(i) *X is complete.*

(ii) *Every mapping T on X satisfying the following has a fixed point:*
- *There exists $r \in [0, 1)$ such that $\theta(r)d(x, Tx) \leq d(x, y)$ implies $d(Tx, Ty) \leq rd(x, y)$ for all $x, y \in X$.*

(iii) *There exists $r \in (0, 1)$ such that every mapping T on X satisfying the following has a fixed point:*
- *$\frac{1}{10000}d(x, Tx) \leq d(x, y)$ implies $d(Tx, Ty) \leq rd(x, y)$ for all $x, y \in X$.*

They said: "The authors are very attracted by $\theta(r)$ because $\theta(r)$ does not seem to be natural. We know $\theta(r)$ is the best constant because of the existence of counterexamples. To find an intuitive reason is another motivation. However, we have not found such a reason yet. On the contrary, we have to raise one problem concerning $\theta(r)$."

Theorem 1 can be extended as follows:

$$\text{(i)} \Longleftrightarrow \text{Theorem H}(\gamma 1) \text{ for metric spaces} \Longleftrightarrow \text{(ii)} \Longleftrightarrow \text{(iii)}.$$

Two generalizations of Theorem 1 to multimaps are obtained:

Theorem 2. *Define a strictly decreasing function η from $[0,1)$ onto $(1/2,1]$ by*

$$\eta(r) = \frac{1}{1+r}.$$

Let (X,d) be a complete metric space and let T be a mapping from X into $CB(X)$. Assume that there exists $r \in [0,1)$ such that

$$\eta(r)d(x,Tx) \le d(x,y) \text{ implies } H(Tx,Ty) \le r\,d(x,y)$$

for all $x,\ y \in X$. Then there exists $z \in X$ such that $z \in Tz$.

New Proof. This follows from our Theorem H(δ1). In fact, if $\{x\} = Tx$ for some $x \in X$, then let $z = x$. For any $x \in X\backslash T(x)$ and any $y \in X\backslash\{x\}$, we have

$$\eta(r)d(x,Tx) = \eta(r)\inf\{d(x,z) : z \in Tx\} \le d(x,y) \text{ implies } H(Tx,Ty) \le r\,d(x,y).$$

Therefore, by Theorem H(δ1), T has a fixed point $z \in X$. □

In this theorem, several things can be improved, that is, metric space, CB, and the Suzuki condition can be replaced by more general ones.

Finally, they added "another result in [45] is generalized as Park and Bae [36] in 1981 generalized the Meir-Keeler fixed point theorem."

Kikkawa and Suzuki [19] in 2008

The following is a Kannan version of the Suzuki theorem [45]:

Theorem 2.2. (Kikkawa-Suzuki) *Let T be a mapping on complete metric space (X,d) and let φ be a non-increasing function from $[0,1)$ into $(1/2,1]$ defined by*

$$\varphi(r) = \begin{cases} 1, & \text{if } 0 \le r \le \frac{1}{\sqrt{2}}, \\ \frac{1}{1+r}, & \text{if } \frac{1}{\sqrt{2}} \le r < 1. \end{cases} \tag{3}$$

Let $\alpha \in [0,1/2)$ and $r = \alpha/(1-\alpha) \in [0,1)$. Suppose that

$$\varphi(r)d(x,Tx) \le d(x,y) \text{ implies } d(Tx,Ty) \le \alpha d(x,Tx) + \alpha d(y,Ty)$$

for all $x,y \in X$. Then, T has a unique fixed point z, and $\lim_n T^n x = z$ holds for every $x \in X$.

This gives an example of RHR maps and Theorem P can be applied.
They said the following theorem shows that $\varphi(r)$ is the best constant for every r:

Theorem 2.4. *Define a function φ as in Theorem 2.2. For every $\alpha \in [0,1/2)$, putting $r = \alpha/(1-\alpha)$, there exist a complete metric space (X,d) and a mapping T on X such that T has no fixed points and*

$$\varphi(r)d(x,Tx) < d(x,y) \text{ implies } d(Tx,Ty) \le \alpha\,d(x,Tx) + \alpha\,d(y,Ty)$$

for all $x,\ y \in X$.

New Proof. Note that $\theta(r)d(x,Tx) < d(x,y)$ means T can not have a fixed point $x = y = Tx$. □

In this paper, it is noted that $d(Tx,Ty) \le r\,d(x,y)$ in the original Suzuki type map in [45] can be replaced by one of the following in Theorem 3.1:

$$d(Tx,Ty) \le r\max\{d(x,Tx),d(y,Ty)\},$$

$$d(Tx,Ty) \le \max\{\alpha d(x,Tx),\beta d(y,Ty)\},\ \ \alpha,\beta \in [0,1).$$

Similarly, Theorem 3.3 is modified applying such conditions.

In fact, they gave lengthy unnecessarily proofs for their theorems.

Kikkawa and Suzuki [20] in 2008

In the previous paper, Kikkawa-Suzuki discussed a similarity between contractions and Kannan mappings. In this paper, they continue to discuss a similarity between contractions and generalized Kannan mappings — M-Kannan mappings.

Suzuki introduced the notion of τ-distance in order to generalize the results of Kada et al., Tataru, Zhong and others. Let (X,d) be a metric space and let p be a τ-distance on X.

The authors defined several mapping classes on X as in the following:

Theorem 8. *Let (X,d) be a metric space. Then*

$$TC_0(X) = TC(X) = TM_0(X) = TM(X)$$

holds.

In the proof, the selfmap T on (X,p) satisfies the following:

$$p(Tx,T^2x) \le r\,p(Tx,x) \text{ and } p(T^2x,Tx) \le r\,p(x,Tx) \text{ for } x \in X,$$

where $r \in [0,1)$.

Note that T is of the RHR type. Therefore our arguments on metric spaces can be applied to several theorems in this paper.

Kikkawa and Suzuki [21] in 2009

The authors give some notes on recent fixed point theorems with constants which are generalizations of the Banach contraction principle.

Let (X,d) be a metric space. Let $\mathrm{CB}(X)$ be the family of all nonempty closed bounded subsets of X. Let $H(\cdot,\cdot)$ be the Hausdorff metric, i.e.,

$$H(A,B) = \max\{\delta(A,B),\delta(B,A)\}\ for\ A,\ B \in \mathrm{CB}(X),$$

where $\delta(A,B) = \sup_{x\in A}\inf_{y\in B} d(x,y)$.

In this paper, $H(A,B)$ in their Nonlinea Analysis paper [18] is replaced by $\delta(A,B)$ and similar results are obtained.

Enjouji, Nakanishi, and Suzuki [10] in 2009

In order to observe the condition of Kannan mappings, the authors prove a generalization of Kannan's fixed point theorem. Their theorem involves constants and they obtain the best constants to ensure a fixed point. They consider "$\alpha d(x,Tx) + \beta d(y,Ty)$" instead of "$\alpha d(x,Tx) + \alpha d(y,Ty)$."

Let $\Delta = \{(\alpha,\beta) : \alpha \ge 0, \beta \ge 0, \alpha+\beta < 1\}$. Define a nonincreasing function $\psi : \Delta \to (1/2,1]$. Let T be a map on a complete metric space such that there exists $\alpha,\beta \in \Delta$ satisfying

$$\psi(\alpha,\beta)d(x,Tx) \le d(x,y) \text{ implies } d(Tx,Ty) \le \alpha d(x,Tx) + \beta d(y,Ty)$$

for all $x,y \in X$.

Note that T is an RHR map and that the condition $\psi(\alpha, \beta)d(x, Tx) < d(x, y)$ in Theorem 4,1 implies nonexistence of a fixed point of T.

Nakanishi and Suzuki [25] in 2010

In this paper, in order to observe the condition of Kannan maps more deeply, they prove a generalization of Kannan's fixed point theorem.

Moreover, they assumed something like $\theta(\alpha)d(x, Tx) < d(x, y)$, which is false for $x = y = Tx$. Hence, from the beginning, such T can not have a fixed point.

Further, for $(\alpha, \beta) \in \Delta = [0, 1)^2$ and a function $\varphi : \Delta \to (1/2, 1]$, let T be a selfmap of a complete metric space (X, d) satisfying

$$\varphi(\alpha, \beta)d(x, Tx) \leq d(x, y) \text{ implies } d(Tx, Ty) \leq \max\{\alpha d(x, Tx), \beta d(y, Ty)\},$$

for all $x, y \in X$.

Note that T is an RHR map and that the condition $\varphi(\alpha, \beta)d(x, Tx) < d(x, y)$ does not hold for $x = y = Tx$.

5. Our previous works on the RHR maps

As the author recalled in [29], he had engaged to study the metric fixed point theory from the late 1970's to the early 1990's. In that period, he recognized a theorem of Rus [39]. In 2022, Professor Szaz reminded me our old form of Metatheorem suggested in [28]. Around that time the author found a very particular form the Hicks-Rhoades theorem [12] from Jachymski's paper [15] in 2003. Then the author called it the Rus-Hicks-Rhoades theorem, found many earlier relatives and wrote a paper [31].

Since then he tried to collect literatures on the RHR maps and found that it had several former names. In fact, V. Berinde [4] in 2003 noted the previous studies on RHR maps: the so called Banach orbital condition

$$d(Tx, T^2x) \leq \alpha\, d(x, Tx), \quad \text{for all } x \in X,$$

studied by various authors in the context of fixed point theorems, see for example Kasahara [17], Hicks and Rhoades [12], Ivanov [14], Rus [38], [40], [41, Example 4.6] and Taskovic [46].

Now we introduce the abstracts of our previous articles which are closely related to the present one:

Park [30] in 2022

We establish Ordered Fixed Point Theory based on our long-standing Metatheorem. This theory is independent from the traditional ones on Analytic, Metric, or Topological Theories. In this article, many well-known fixed point theorems on ordered sets are equivalently formulated to existence theorems on maximal (resp. minimal) elements, common fixed points, common stationary points, and others. Hence, we obtain characterizations of well-known results due to Zermelo, Zorn, Knaster-Tarski, Tarski-Kantorovitch, Nadler, Ekeland, Caristi, Brézis-Browder, Takahashi, Lin-Du, and many others in their improved or equivalent formulations. Consequently, we obtain scores of new order theoretic theorems based on the Brøndsted-Jachymski Principle recently due to ourselves and its generalized 2023 Metatheorem.

In this article, Theorem H was first given.

Park [31] in 2023

Let (X, d) be a complete metric space and $f : X \to X$ a map satisfying $d(fx, f^2x) \leq \alpha d(x, fx)$ for every $x \in X$, where $0 < \alpha < 1$. The fixed point theorems due to Rus (1973) and Hicks-Rhoades (1979) on such maps were extended or improved by Park (1980), Harder-Hicks-Saliga (1993), Suzuki (2001), and

Jachymski (2003). Moreover, fixed point theorems of Zermelo (1904), Banach (1922) and Caristi (1976), extended versions for multimaps due to Nadler (1969) and Covitz-Nadler (1970) are also closely related to the Rus-Hicks-Rhoades theorem. Finally, we unify these based on a particular form of our 2023 Metatheorem.

In this paper, the Rus-Hicks-Rhoades (RHR) map is first introduced and the Hicks-Rhoades theorem is still incorrectly quoted.

Park [32] in 2023

Let (X, d) be a quasi-metric space. A Rus-Hicks-Rhoades (RHR) map $f : X \to X$ is the one satisfying $d(fx, f^2x) \leq \alpha d(x, fx)$ for every $x \in X$, where $\alpha \in [0, 1)$. In our previous work [37], we collected various fixed point theorems closely related to RHR maps. In the present article, we collect almost all things we know about the RHR maps and their examples. Moreover we derive new classes of generalized RHR maps and fixed point theorems on them. Consequently, many of known results in metric fixed point theory are improved and reproved in easy way.

In this paper, the author based on literature not sufficiently enough and the Hicks-Rhoades theorem was still incorrectly stated. However, old forms of Theorems H and P were given. Moreover, simple proofs of works of Suzuki and his colleagues were given based on Theorem H(γ1).

Park [33] in 2023

In our previous work entitled "Foundations of Ordered Fixed Point Theory"[30] in 2022, we established a large number of improved versions of historically well-known maximality theorems and fixed point theorems related to order structure. It is based on our long-standing Metatheorem and the Brøndsted-Jachymski principle established by ourselves in 2022. In fact, our Metatheorem states that some well-known order theoretic fixed point theorems are equivalently formulated to existence theorems on maximal (or minimal) elements, common fixed points, common stationary points, and others. In the present article, we collect and introduce the contents of articles related to the Metatheorem. Now Ordered fixed point theory will be a rich source of researches comparable to Analytical, Metric, and Topological fixed point theory.

In this historical article, new versions of Theorems H and P were given and a short proof of Suzuki's 2008 theorem was given.

Park [34] in 2023

Our aim in this article is to show some well-known theorems on metric spaces also hold for quasi-metric spaces (without symmetry) from the beginning. We check this fact for the Banach contraction principle, the Covitz-Nadler fixed point theorem, the Rus-Hicks-Rhoades fixed point theorem, and others. In these theorems the concepts of continuity and completeness can be replaced by orbital continuity and T-orbital completeness for a selfmap T, respectively. Consequently, we improve and generalize the basic known theorems in the metric fixed point theory.

Suzuki's 2008 theorem was given as an example.

Park [35] in 2023

Our aim in this article is to show that certain well-known theorems on metric spaces hold for quasi-metric spaces (without symmetry) from the beginning. We check this claim for theorems of Banach, Ekeland, Caristi, Takahashi, Rus-Hicks-Rhoades, and others. Moreover our Brøndsted-Jachymski principle – on the relation among maximal elements, fixed elements, and periodic elements of partially ordered quasi-metric spaces – improves known fixed point theorems. Consequently, we extend many theorems in the metric or ordered fixed point theory by adopting quasi-metric instead of metric.

4. Epilogue

In our previous works [34], [35] and the present one, we showed metric fixed point theorems hold for quasi-metric spaces from the beginning.

Even for the Banach contraction principle, certain traditional monographs or text-books on fixed point theory or general topology stated for metric spaces only, for example, Dugundji [8], Willard [47], Smart [42], Istratescu [13], Dugundji-Granas [9], Goebel-Kirk [11], Kirk [23] and many others. All of them stated the principle for metric spaces only, but their proofs do not use the symmetry of a metric. Of course, they do not mention the concept of quasi-metric spaces.

Beginning from our [30] in 2022, we have published nearly two dozen articles on our 2023 Metatheorem and related topics. These could add up many new results to Ordered Fixed Point Theory in [30]. Moreover, our Metatheorem has numerous applications; for recent examples, see [32]-[35].

In the present article, we corrected only some of inaccurate statements in Metric Fixed Point Theory. However there are many results related to the Rus-Hicks-Rhoades theorem due to scores of other authors. Since the theorem and its extensions properly include the corresponding ones of the Banach contraction, the future study on Metric Fixed Point Theory would be concentrated to extend the theorem and its new applications.

There are thousands of artificial generalizations of metric spaces. Most of them assume the symmetry which might be eliminated as in the present article.

References

[0] 1 H. Aydi, M. Jellali, E. Karapinar, *On fixed point results for α-implicit contractions in quasi-metric spaces and consequences*, Nonlinear Anal. Model. Control, 21(1) (2016) 40–56.

[0] 2 J.S. Bae, *Fixed point theorems for weakly contractive multivalued maps*, J. Math. Anal. Appl. 284 (2003) 690–697.

[1] J.S. Bae, E.W. Cho, S.H. Yeom, *A generalization of the Caristi-Kirk fixed point theorem and its applications to mapping theorems*, J. Korean Math. Soc. 31 (1994) 29–48.

[2] V. Berinde, *On the approximation of fixed points of weak contractive mappings*, Carpathian J. Math. 19(1) (2003) 7– 22.

[3] V. Berinde, A. Petrusel, I. Rus, *Remarks on the terminology of the mappings in fixed point iterative methods in metric spaces*, Fixed Point Theory, 24(2) (2023) 525–540. DOI: 10.24193/fpt-ro.2023.2.05

[4] S. Cobzaş, *Fixed points and completeness in metric and generalized metric spaces*, Jour. Math. Sci. 250(3) (2020) 475–535. DOI 10.1007/s10958-020-05027-1

[5] H. Covitz, S.B. Nadler, Jr. *Multi-valued contraction mappings in generalized metric spaces*, Israel J. Math. 8 (1970) 5–11.

[6] J. Dugundji, *Topology*, Allyn and Bacon, Boston, 1966.

[7] J. Dugundji, A. Granas, *Fixed Point Theory I*, Polish Scientific Publishers, Warszawa, 1982.

[8] Y. Enjouji, M. Nakanishi, T. Suzuki, *A generalization of Kannan's fixed point theorem*, Fixed Point Theory Appl. 1 (2009) 192872, 10pp.

[9] K. Goebel, W.A. Kirk, *Topics in metric fixed point theory*, Cambridge Univ. Press 1990.

[10] T.L. Hicks, B.E. Rhoades, *A Banach type fixed point theorem*, Math. Japon. 24 (1979) 327–330.

[11] V.I. Istratescu, *Fixed Point Theory*, D. Reidel Pub. Co. 1981

[12] A.A. Ivanov,*Fixed points of metric space mappings* (in Russian), Isledovaniia po topologii. II, Akademia Nauk, Moskva, 1976, pp. 5–102.

[13] J. Jachymski, *Converses to fixed point theorems of Zermelo and Caristi*, Nonlinear Analysis 52 (2003) 1455–1463.

[14] M. Jleli, B. Samet, *Remarks on G-metric spaces and fixed point theorems*, Fixed Point Theory Appl. 2012:210, 2012.

[15] S. Kasahara, *On some generalizations of the Banach contraction theorem*, Publ. RIMS, Kyoto Univ. 12 (1976) 427–437.

[16] M. Kikkawa, T. Suzuki, *Three fixed point theorems for generalized contractions with constants in complete metric spaces*. Nonlinear Analysis 69 (2008) 2942–2949.

[17] M. Kikkawa, T. Suzuki, *Some similarity between contractions and Kannan mappings*, Fixed Point Theory Appl. (2008) 649749, 8. doi: 10.1155/2007/49749

[18] M. Kikkawa, T. Suzuki, *Some similarity between contractions and Kannan mappings, II*, Bull. Kyushu Inst. Tech. Pure Appl. Math. 55 (2008) 1–13.

[19] M. Kikkawa, T. Suzuki, *Some notes on fixed point theorems with constants*, Bull. Kyushu Inst. Tech. Pure Appl. Math. 56 (2009) 11–18.

[20] W.A. Kirk, *Caristi's fixed point theorem and metric convexity*, Colloquium Mathematicum, 36(1) (1976) 81–86.

[21] W.A. Kirk, *Contraction mappings and extensions*, Chapter 1, Handbook of Metric Fixed Point Theory (W.A. Kirk and B. Sims, eds.), Kluwer Academic Publ. (2001) 1–34.

[22] S.B. Nadler, Jr. *Multi-valued contraction mappings*, Pacific J. Math. 30 (1969) 475-488.
[23] M. Nakanishi1, T. Suzuki, *An observation on Kannan mappings*, Cent. Eur. J. Math. 8(1) (2010) 170–178. DOI: 10.2478/s11533-009-0065-9
[24] S. Park, *On general contractive-type conditions*, J. Korean Math. Soc. 17 (1980) 131–140.
[25] S. Park, *Characterizations of metric completeness*, Colloquium Mathematicum 49(1) (1984) 21–26,
[26] S. Park, *On generalizations of the Ekeland type variational principles*, Nonlinear Anal. TMA 39 (2000) 881–889.
[27] S. Park, *Forty five years with fixed point theory*, J. Fixed Point Theory (2019) Article ID 1: 1–25.
[28] S. Park, *Foundations of ordered fixed point theory*, J. Nat. Acad. Sci., ROK, Nat. Sci. Ser. 61(2) (2022) 1–51.
[29] S. Park, *Relatives of a theorem of Rus-Hicks-Rhoades*, Letters Nonlinear Anal. Appl. 1 (2023) 57–63. ISSN 2958-874X
[30] S. Park, *Almost all about Rus-Hicks-Rhoades maps in quasi-metric spaces*, Adv. Th. Nonlinear Anal. Appl. 7(2) (2023) 455–471. DOI 0.31197/atnaa.1185449
[31] S. Park, *History of the metatheorem in ordered fixed point theory*, J. Nat. Acad. Sci., ROK, Nat. Sci. Ser. 62(2) (2023)
[32] S. Park, *All metric fixed point theorems hold for quasi-metric spaces*, Manuscript.
[33] S. Park, *The use of quasi-metric in the metric fixed point theory*, Manuscript.
[34] S. Park, J.S. Bae, *Extension of a fixed point theorem of Meir and Keeler*, Arkiv för Mat. 19 (1981), 223–228. MR 83e:54048. Zbl 483.47040.
[35] S. Park, B.E. Rhoades, *Comments on characterizations for metric completeness*, Math. Japon. 31(1) (1986) 95–97.
[36] I.A. Rus, *The method of successive approximations* (in Russian), Revue Roum. de Math. Pures et Appl. 17 (1972) 1433–1437.
[37] I.A. Rus, *Teoria punctului fix, II*, Univ. Babes-Bolyai, Cluj, 1973.
[38] I.A. Rus, *Principles and Applications of the Fixed Point Theory* (in Romanian), Editura Dacia, Cluj-Napoca, 1979.
[39] I.A. Rus, *Picard operators and applications*, Scientiae Math. Japon. 58(1) (2003) 191–219.
[40] D.R. Smart, *Fixed point theorems*, Cambridge Univ. Press, 1974.
[41] T. Suzuki, *Generalized distance and existence theorems in complete metric spaces*, J. Math. Anal. Appl. 253 (2001) 440–458.
[42] T. Suzuki, *Generalized Caristi's fixed point theorems by Bae and others*, Jour. Math. Anal. Appl. 302(2) (2005) 502–508.
[43] T. Suzuki, *A generalized Banach contraction principle that generalizes metric completeness*, Proc. Amer. Math. Soc. 136(5) (2008) 1861–1869.
[44] M. Taskovic, *Osnove teorije fiksne tacke (Fundamental Elements of Fixed Point Theory)*, Matematicka biblioteka 50, Beograd, 1986.
[45] S. Willard, *General Topology*, Addison-Wesley, 1970.