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Existence and uniqueness result for fractional dynamic equations on metric-like spaces

İnci M. Erhan[a], Najeh Redjel[b]

[a]Department of Computer Engineering, Aydın Adnan Menderes University, 09010, Aydın, Türkiye
[b]Department of Mathematics, University of Souk-Ahras, Souk-Ahras, Algeria

Abstract

The problem of existence and uniqueness of solutions of initial value problems associated with a nonlinear fractional dynamic equation of Caputo type on an arbitrary time scale of order $\alpha > 0$ is stated as a fixed point problem on a metric-like space. The initial conditions are assummed to be homogeneous. A theorem on the existence and uniqueness of a solution of the problem is stated and proved. Examples on two different time scales verifying the theoretical findings are presented and numerical computation of several initial terms of the iterative sequence of approximations is included.

Keywords: time scale, Caputo fractional dynamic equation, fixed point, metric-like
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1. Introduction

Fractional differential calculus remains as one of the most studied areas of applied mathematics due to its extensive application potential [18, 19, 20, 21]. In addition to the fractional derivatives of Riemann-Liouville and Caputo type, new types of fractional derivatives have been defined and various real life problems have been modelled by using these new types of derivatives [5, 16].

The concept of fractional derivative has also been carried on to time scales [4, 6, 13]. The dynamic equations which succesully unify the differential and difference equations, have been extended to fractional dynamic equations of Riemann-Liouville and Caputo type and investigated from different aspects by many

Email addresses: incierhan@gmail.com (İnci M. Erhan), najehredjel@yahoo.fr, n.radjal@univ-soukahras.dz (Najeh Redjel)

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authors [3, 12, 17]. In a very recent book by Georgiev [13], the fractional dynamic calculus and fractionl dynamic equations have been introduced in a detailed and complete way.

Also recently, the problem of existence and uniqueness for fractional dynamic equations and associated initial and boundary value problems have been studied and some results appeared in the literature [15, 23, 11].

In this study, we discuss the existence and uniqueness of an initial value problem for a fractional dynamic equation of Caputo type of arbitrary order with homogeneous initial conditions, given as

$$\begin{cases} {}^{C}D^{\alpha}_{\Delta,t_0}u(t) &= f(t,u(t)), \quad t\in T, \\ {}^{C}D^{k}_{\Delta,t_0}u(t) &= 0, \quad k\in\{0,\dots,m-1\}, \end{cases} \tag{1}$$

where $\alpha>0$, $m=-\lceil -\alpha\rceil$, $f:\mathbb{T}\times\mathbb{R}\to\mathbb{R}$ is a given function and $T=[a,b]_{\mathbb{T}}$ is a time scale interval containing t_0. By converting the problem into an integral equation of Volterra type, we state the problem as a fixed point problem for the associated integral operator. Employing the fixed point results on metric-like spaces, we prove an existence and uniqueness theorem for the solution of the initial velue problem.

The presentation is organizated as follows. Next section is devoted to the definition of preliminary notions and the fractional derivative on arbitrary time scale. In section 3, the metric-like spaces and the relevant theoretical results are introduced. The existence and uniqueness of the initial value problem (1) is studied in Section 4 and followed by numerical examples. Conlusion is given in the last section.

2. Brief discussion on fractional derivative on time scales

We start with the presentation of the main concepts which are involved in the definition of fractional derivative on time scales. For the introductory information and the basics of calculus on time scales we refer the reader to [7, 8, 9].

The fractional derivative on time scale have been defined in two different ways: by means of the usual power function, that is, $h_\alpha(t,s)=(t-s)^\alpha$, see [3], and by means of the generalized power functions $h_\alpha(t,s)$ whose form depends on the time scale, see [13].

In this study, we employ the second form which is more general.

On the rest of the section, $\mathbb{T}$ will denote an arbitrary time scale, σ, μ and Δ will denote the forward jump operator, graininess function and delta differential operator, respectively.

The time scale monomials will be denoted and defined as

$$h_0(t,s)=1, \quad h_n(t,s)=\int_s^t h_{n-1}(t,y)\Delta y, \quad n\in\mathbb{N}_0.$$

Next, we recall the definitions of regressive functions and the generalized exponential function on time scales.

Definition 2.1. *[9]*

1. *A function $g:\mathbb{T}\to\mathbb{R}$ is called regressive if*

$$1+\mu(t)g(t)\neq 0 \quad \textit{for all} \quad t\in\mathbb{T}^\kappa.$$

2. *The set of all regressive functions on a time scale $\mathbb{T}$ is denoted by $\mathcal{R}$.*
3. *For a function $g\in\mathcal{R}$, the generalized exponential function is defined and denoted by*

$$e_g(t,s)=e^{\int_s^t \frac{1}{\mu(\tau)}\mathrm{Log}(1+\mu(\tau)g(\tau))\Delta\tau} \quad \textit{for} \quad s,t\in\mathbb{T}.$$

where Log *is the principal logarithm function.*

Detailed information on the generalized exponential function and its properties can be found in [7, 9]. We also recall the definitons of the Laplace transform, shift and convolution on time scales.

Definition 2.2. *[13] Let $\mathbb{T}_0$ be a time scale containing 0 and for which $\sup \mathbb{T}_0 = \infty$.*
The Laplace transform of the function $g : \mathbb{T}_0 \to \mathbb{C}$ is defined as

$$\mathcal{L}(g)(w) = \int_0^\infty g(y) e^\sigma_{\ominus w}(y, 0)\Delta y, \tag{2}$$

for all $w \in \mathcal{D}(g)$, where $\mathcal{D}(g)$ is the set

$$\begin{aligned}\mathcal{D}(g) \;=\; \{w \in \mathbb{C} :\; & 1 + w\mu(t) \neq 0 \text{ for all } t \in \mathbb{T}_0 \\ & \text{and the improper integral } \int_0^\infty g(y)e^\sigma_{\ominus w}(y,0)\Delta y \text{ exists} \Big\}.\end{aligned}$$

Note that here we used the notation $\ominus w = -\frac{w}{1+w\mu(t)}$ and $f^\sigma = f \circ \sigma$ on time scales.

Definition 2.3. *[13] Let $g : \mathbb{T} \to \mathbb{C}$. The shift (delay) of g is denoted by $\widehat{g}$ and is the solution of the shifting problem*

$$\begin{cases} v^{\Delta_t}(t, \sigma(s)) = -v^{\Delta_s}(t, s), & t \in \mathbb{T}, \quad t \geq s \geq t_0, \\ v(t, t_0) = g(t), & t \in \mathbb{T}, \quad t \geq t_0. \end{cases} \tag{3}$$

Definition 2.4. *[13] The convolution $f * g$ of the functions $f, g : \mathbb{T} \to \mathbb{C}$ is defined as*

$$(f * g)(t) = \int_{t_0}^t \widehat{f}(t, \sigma(s))g(s)\Delta s, \quad t \in \mathbb{T}, \quad t \geq t_0. \tag{4}$$

The Laplace transform, shift and convolution are explained in details in [13].

Finally, we define the generalized and fractional generalized Δ-power functions on time scales. These notions are defined on a time scale of the form

$$\mathbb{T} = \{t_n :\; n \in \mathbb{N}_0\},$$

where

$$\begin{aligned}&\lim_{n\to\infty} t_n = \infty, \\ &\sigma(t_n) = t_{n+1}, n \in \mathbb{N}_0, \\ &\inf_{n\in\mathbb{N}_0} \mu(t_n) > 0.\end{aligned}$$

Definition 2.5. *[13] Let $\alpha \in \mathbb{R}$*

1. *The generalized Δ-power function $h_\alpha(t, t_0)$ on $\mathbb{T}$ is defined as*

$$h_\alpha(t, t_0) = \mathcal{L}^{-1}\left(\frac{1}{z^{\alpha+1}}\right)(t), \quad t \geq t_0,$$

   for all $z \in \mathbb{C}\backslash\{0\}$ such that $\mathcal{L}^{-1}$ exists.

2. *The fractional generalized Δ-power function $h_\alpha(t, s)$ on $\mathbb{T}$ is defined as the shift of $h_\alpha(t, t_0)$, that is,*

$$h_\alpha(t, s) = \widehat{h_\alpha(\cdot, t_0)}(t, s), \quad t, s \in \mathbb{T}, \quad t \geq s \geq t_0.$$

From the definition, the fractional generalized Δ-power functions on the time scales $\mathbb{R}$ and $\mathbb{Z}$ are obtained as follows.

Example 2.6. *[13]*

1. *Let $\mathbb{T} = \mathbb{R}$. Then*

$$h_\alpha(t, s) = \frac{(t - s)^\alpha}{\Gamma(\alpha + 1)}, \quad t \geq s.$$

2. *Let* $\mathbb{T} = \mathbb{N}_0$. *Then*

$$\begin{aligned} h_\alpha(t,s) &= \frac{(t-s)^{(\alpha)}}{\Gamma(\alpha+1)} \\ &= \frac{\Gamma(t-s+1)}{\Gamma(\alpha+1)\Gamma(t-s+1-\alpha)}, \quad t \geq s. \end{aligned}$$

It remains to define the Riemann-Liouville and Caputo fractional Δ-integral and Δ-derivative on a time scale $\mathbb{T}$ having the form given above. As mentioned above, these definitions require the definition of generalized Δ-power function.

Definition 2.7. *[13] Let* $\alpha \geq 0$ *denotes the order of the derivtive and* $m = -\overline{[-\alpha]}$, *that is, the integer part of* $-\alpha$. *Let* $g : \mathbb{T} \to \mathbb{R}$.

1. *The Riemann-Liouville fractional* Δ*-integral of order* α *for the function* g *is defined as*

$$\begin{aligned} (I^0_{\Delta,t_0} g)(t) &= g(t), \\ (I^\alpha_{\Delta,t_0} g)(t) &= (h_{\alpha-1}(\cdot,t_0) * g)(t) \\ &= \int_{t_0}^t \widehat{h_{\alpha-1}(\cdot,t_0)}(t,\sigma(u))g(u)\Delta u \\ &= \int_{t_0}^t h_{\alpha-1}(t,\sigma(u))g(u)\Delta u, \quad \alpha > 0, \quad t \geq t_0. \end{aligned} \tag{5}$$

2. *For* $s,t \in \mathbb{T}^{\kappa^m}$, $s < t$ *the Riemann-Liouville fractional* Δ*-derivative of order* α *is defined as*

$$D^\alpha_{\Delta,s} g(t) = D^m_\Delta I^{m-\alpha}_{\Delta,s} g(t), \quad t \in \mathbb{T}, \tag{6}$$

   if it exists.

3. *For* $\alpha < 0$, *we define*

$$\begin{aligned} D^\alpha_{\Delta,s} g(t) &= I^{-\alpha}_{\Delta,s} g(t), \quad t,s \in \mathbb{T}, \quad t > s. \\ I^\alpha_{\Delta,s} g(t) &= D^{-\alpha}_{\Delta,s} g(t), \quad t,s \in \mathbb{T}^{\kappa^r}, \quad t > s, \quad r = \overline{[-\alpha]} + 1. \end{aligned} \tag{7}$$

Remark 2.8. *If we note that the generalized* Δ *power function* $h_\alpha(t,t_0)$ *on the set of real numbers* $\mathbb{R}$ *is*

$$h_\alpha(t,t_0) = \frac{(t-t_0)^\alpha}{\Gamma(\alpha+1)}, \quad t \geq t_0,$$

we observe that if $\mathbb{T} = \mathbb{R}$, *that is, if* Δ *derivative is replaced by the classical derivative, the Riemann-Liouville fractional* Δ*-derivative defined in* (6) *becomes the usual Riemann-Liouville fractional derivative.*

Finally, we define Caputo fractional Δ-derivative in terms of the previous definitions as follows.

Definition 2.9. *[13] For a function* $g : \mathbb{T} \to \mathbb{R}$ *the Caputo fractional* Δ*-derivative of order* α *is denoted by* ${}^C D^\alpha_{\Delta,t_0}$ *and defined via the Riemann-Liouville fractional* Δ*-derivative of order* α *as follows*

$${}^C D^\alpha_{\Delta,t_0} = D^\alpha_{\Delta,t_0}\left(g(t) - \sum_{k=0}^{m-1} h_k(t,t_0) g^{\Delta^k}(t_0)\right), \quad t > t_0, \tag{8}$$

where $m = \overline{[\alpha]} + 1$ *if* $\alpha \notin \mathbb{N}$ *and* $m = \overline{[\alpha]}$ *if* $\alpha \in \mathbb{N}$.

The following theorem provides an alternative representation of the Caputo fractional Δ-derivative (see Theorem 7.1 in [13]).

Theorem 2.10. *Let* $\alpha > 0$, $m = \overline{[\alpha]} + 1$ *if* $\alpha \notin \mathbb{N}$ *and* $m = \alpha$, *if* $\alpha \in \mathbb{N}$.

1. *If* $\alpha \notin \mathbb{N}$ *then*

$${}^C D^\alpha_{\Delta,t_0} g(t) = I^{m-\alpha}_{\Delta,t_0} D^m_{\Delta,t_0} g(t), \quad t \in \mathbb{T}, t > t_0.$$

2. *If* $\alpha \in \mathbb{N}$ *then*

$${}^C D^\alpha_{\Delta,t_0} g(t) = g^{\Delta^m}(t), \quad t \in \mathbb{T}, t > t_0.$$

Remark 2.11. *Regarding the result of the Theorem 2.10, if* $\mathbb{T} = \mathbb{R}$, *the Caputo fractional* Δ*-derivative defined in* (8) *becomes the usual Caputo fractional derivative.*

3. Review on metric like spaces

In this section, we recall the basics on metric-like spaces. The initial value problem (1) will be stated as a fixed point problem in the framework of metric-like space. Accordingly, we need to review the concept of metric-like and some properties of metric-like spaces.

Metric-like spaces have been defined initially by Amini-Hanrandi [1] and later, used by several authors to introduce a new classes of generalized contractive mappings in order to study the existence and uniqueness of the solution of certain differential and integral equations (see [22, 14, 2]). The metric-like and the metric-like space are defined as follows.

Definition 3.1. *[1]. Let X be a nonempty set. A a function $\rho : X \times X \to [0, \infty)$ is a metric-like on X, if it satisfies the following conditions for all $v, u, w \in X$.*

(ρ_1) *If $\rho(v, u) = 0$ then $v = u$;*

(ρ_2) *$\rho(v, u) = \rho(u, v)$;*

(ρ_3) *$\rho(v, u) \leq \rho(v, w) + \rho(w, u)$.*

The pair (X, ρ) is then called a metric-like space.

A thorough information on metric-like spaces can be found in [1]. Some metric-like spaces are presented in the next example.

Example 3.2. *[22]. Let $\rho_j : \mathbb{R} \times \mathbb{R} \longrightarrow [0, +\infty)$, for $j = 1, 2, 3$ be functions defined as follows.*

1. $\rho_1(v, u) = |v| + |u| + a$, $a \geq 0$;
2. $\rho_2(v, u) = |v - b| + |u - b|$, $b \in \mathbb{R}$;
3. $\rho_3(v, u) = v^2 + u^2$.

Then ρ_1, ρ_2 and ρ_3 are all metric-likes on $\mathbb{R}$.

Remark 3.3. *Observe that $\rho_j(v, v)$, $j = 1, 2, 3$ in the previous Example may not be 0 for $v \in \mathbb{R}$, and consequently metric-like spaces are not metric spaces, in general.*

The following concepts were introduced by [1] in the setting of metric-like spaces.

- Each metric-like ρ on X generates a topology on X whose base is the family of open ρ balls

$$B_\rho(v, \epsilon) = \{u \in X : |\rho(v, u) - \rho(v, v)| < \epsilon\}, \text{ for all } v \in X \text{ and } \epsilon > 0.$$

- A sequence $(v_n)_{n \geq 0}$ in metric-like space (X, ρ) converges to $v \in X$ if and only if

$$\lim_{n \longrightarrow \infty} \rho(v_n, v) = \rho(v, v).$$

- A sequence $(v_n)_{n \geq 0} \subset X$ is called ρ-Cauchy sequence if

$$\lim_{n,m \longrightarrow \infty} \rho(v_n, u_m) \text{ exists and is finite.}$$

- A metric-like space (X, ρ) is said to be complete if for each ρ-Cauchy sequence $(v_n)_{n \geq 0}$, there exists $v \in X$ such that

$$\lim_{n \longrightarrow \infty} \rho(v_n, v) = \rho(v, v) = \lim_{n,m \longrightarrow \infty} \rho(v_n, v_m).$$

Remark 3.4. *The fact that in metric-like spaces, the limit of a convergent sequence is not necessarily unique, is demonstrated by an example in [1].*

The following result was established in [1](Theorem 2.11) and provides the existence and uniqueness of fixed point for contractive mappings defined on complete metric-like spaces.

Theorem 3.5. *[1]. Let (X, ρ) be a complete metric-like space and let $T : X \longrightarrow X$ be a mapping satisfying*

$$\rho(Tv, Tu) \leq \alpha(\rho(v, u))\rho(v, u),$$

for all $v, u \in X$ with $v \neq u$ where $\alpha : (0, \infty) \to (0, 1)$ is a nonincreasing functon. Then T has a unique fixed point in X.

In our studies we make use of it for the particular choice of α, that is, $\alpha(t) = k$ with $0 < k < 1$.

4. Existence-uniqueness of Cauchy problems with Caputo fractional derivative on time scales

This section contains the main result and its applications to specific examples. For the rest of the discussion, we suppose that $\mathbb{T}$ is a time scales with forward jump operator σ, delta derivative operator Δ, and delta differentiable graininess function μ. In addition, we assume that $\mathbb{T}$ has the form

$$\mathbb{T} = \{t_n : \ n \in \mathbb{N}_0\},$$

where

$$\lim_{n\to\infty} t_n = \infty,$$
$$\sigma(t_n) = t_{n+1}, n \in \mathbb{N}_0,$$
$$\inf_{n\in\mathbb{N}_0} \mu(t_n) > 0.$$

We also assume that T is a closed interval on the time scale $\mathbb{T}$ having the form $T = [a, b]_{\mathbb{T}}$. As we stated in the introduction, we consider the initial value problem (1) for a fractional dynamic equation with Caputo fractional Δ-derivative given as

$$\begin{cases} {}^{C}D^{\alpha}_{\Delta,t_0}u(t) &=& f(t, u(t)), \quad t \in T, \\ {}^{C}D^{k}_{\Delta,t_0}u(t) &=& 0, \quad k \in \{0, \dots, m-1\}, \end{cases}$$

Here ${}^{C}D^{\alpha}_{\Delta,t_0}$ denotes the Caputo fractional Δ-derivative, $\alpha > 0$ and $m = -\overline{[-\alpha]}$. Also, $f : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ is a given function and $T = [a, b]_{\mathbb{T}}$ is a time scale interval containing t_0.

The right-dense absolutely continuous functions on time scales are defined as follows.

Definition 4.1. *[13] Let J be a time scale interval and $f : J \to \mathbb{R}$. If for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\sum_{i=1}^{n} |f(c_i) - f(d_i)| < \varepsilon,$$

whenever a disjoint collection of time scale intervals $[c_i, d_i) \subset J, i = 1, \dots n$ satisfies

$$\sum_{i=1}^{n} (d_i - c_i) < \delta,$$

then the function f is called right-dense absolutely continuous on the time scale interval J and we write $f \in AC_{\Delta}(J)$. If in addition, $f^{\Delta^l} \in AC_{\Delta}(J)$ for every $l = 0, 1, \dots, k$ and some fixed $k \in \mathbb{N}_0$, then $f \in AC^k_{\Delta}(J)$.

The initial value problem (1) can be written as an integral equation. This conversion is given in the next theorem and its proof can be found in [13].

Theorem 4.2. *[13] Let $D \subset \mathbb{R}$ be an open set, $T = [a,b]_{\mathbb{T}}$ be a time scale interval containing t_0 and let $f : T \times D \to \mathbb{R}$ be a function such that for all $u \in D$, $f(.,u) \in AC_\Delta(T)$. Then $u \in AC^m_\Delta(T)$ is a solution of the problem (1) if and only if it is a solution of the Volterra integral equation*

$$u(t) = \int_{t_0}^{t} h_{\alpha-1}(t,\sigma(\tau))f(\tau,u(\tau))\Delta\tau. \tag{9}$$

We will prove an existence and uniqueness theorem for the solution of the initial value problem (1) in the setting of metric-like spaces.

Let $X = AC_\Delta(T)$ and propose a metric-like ρ on X of the form

$$\rho(v,u) = \sup_{t\in T}(|v(t)| + |u(t)|). \tag{10}$$

Then (X,ρ) is a complete metric-like space.

Define a mapping S as

$$Su(t) = \int_{t_0}^{t} h_{\alpha-1}(t,\sigma(\tau))f(\tau,u(\tau))\Delta\tau. \tag{11}$$

If $f(t,u(t)) \in AC_\Delta(T)$ then the right hand side of the integral equation is a function in $AC_\Delta(T)$. Thus, $S : AC_\Delta(T) \to AC_\Delta(T)$ holds. Then a solution of the problem (1) is a fixed point of S.

Now we state and prove an existence-uniqueness theorem for the solution of the problem (1).

Theorem 4.3. *Let $D \subset \mathbb{R}$ be an open set, $T = [a,b]_{\mathbb{T}}$ be a time scale interval containing t_0 and let the function $f : T \times D \to \mathbb{R}$ be absolutely continuous, that is, $f(t,u(t)) \in AC_\Delta(T)$ and $u \in AC^m_\Delta(T)$. Assume also that*

$$|h_{\alpha-1}(t,\sigma(\tau))| \leq M, \tag{12}$$

for some $M > 0$ and that f satisfies the condition

$$|f(t,u(t))| \leq L|u(t)|, \text{ for all } t \in T \text{ and } u \in D. \tag{13}$$

where $0 < 2ML(b - t_0) < 1$. Then the Cauchy problem (1) has a unique solution.

Proof. We consider the map S defined in (11). If the function $f(t,u(t)) \in AC_\Delta(T)$ then it is Δ-integrable and the right hand side of the integral equation is an absolutely continuous function. Thus, S is a self map on $AC_\Delta(T)$. Using the conditions (12) and(13), we make the following estimate for any $v \in AC^m_\Delta(T)$.

$$\begin{aligned} |Sv(t)| &= \left| \int_{t_0}^{t} h_{\alpha-1}(t,\sigma(\tau))f(\tau,v(\tau))\Delta\tau \right| \\ &\leq \int_{t_0}^{t} |h_{\alpha-1}(t,\sigma(\tau))|\, |f(\tau,v(\tau))|\Delta\tau \\ &\leq M\int_{t_0}^{t} |f(\tau,v(\tau))|\Delta\tau \\ &\leq ML\int_{t_0}^{t} |v(\tau)|\Delta\tau. \end{aligned}$$

This implies that for $u, v \in AC^m_\Delta(T)$,

$$|Sv(t)| + |Su(t)| \leq 2ML\int_{t_0}^{t} |u(\tau)| + |v(\tau)|\Delta\tau,$$

Taking supremum of both sides we conclude

$$\sup_{t\in T}(|Sv(t)| + |Su(t)|) \leq 2ML(b - t_0)\sup_{t\in T}(|v(t)| + |u(t)|),$$

or by means of metric-like,

$$\rho(Sv(t), Su(t)) \leq \lambda\rho(v(t), u(t)),$$

with $0 < \lambda = 2ML(b - t_0) < 1$, from which it follows that the map S defined in (11) satisfies the conditions of the Theorem 3.5 and hence, has a unique fixed point, that is, the initial value problem (1) has a unique solution. □

We apply the result of Theorem 4.3 to specific examples in order to confirm the existence and the uniqueness of solution. In addition, we compute few terms of the sequence of Picard iterations in order to observe its behaviour.

Example 4.4. *Let $\mathbb{T} = \mathbb{N}$ and $T = [t_0, b] = [1, 10] = \{1, 2, 3, ..., 10\}$ be a time scale interval of $\mathbb{T}$. Consider the Cauchy problem*

$$\begin{cases} {}^C D^{3/4}_{\Delta,1} u(t) &= \dfrac{te^{-t}}{40} \sin u(t), \quad t \in T, \\ {}^C D^{0}_{\Delta,1} u(t) &= u(1) = 0. \end{cases} \tag{14}$$

We have $\alpha = \frac{3}{4}$ and $m = -\overline{[-\frac{3}{4}]} = 0$. On this time scale the fractional generalized Δ-power function is given as [13],

$$h_\alpha(t, s) = \frac{\Gamma(t - s + 1)}{\Gamma(\alpha + 1)\Gamma(t - s + 1 - \alpha)}$$

so that we have

$$h_{\alpha-1}(t, s) = h_{-1/4}(t, s) = \frac{\Gamma(t - s + 1)}{\Gamma(\frac{3}{4})\Gamma(t - s + \frac{5}{4})}.$$

Then, using the fact that on the time scale $\mathbb{N}$, the graininess function is $\mu(t) = 1$ and

$$\int_a^b g(y)\Delta y = \sum_{k=a}^{b-1} g(k)\mu(k) = \sum_{k=a}^{b-1} g(k),$$

we compute

$$\begin{aligned} \int_0^t |h_{-1/4}(t, \sigma(\tau))|\Delta\tau &\leq \int_0^{10} |h_{-1/4}(t, \tau + 1)|\Delta\tau \\ &= \int_0^{10} \left| \frac{\Gamma(t - \tau)}{\Gamma(\frac{3}{4})\Gamma(t - \tau + \frac{1}{4})} \right| \Delta\tau \\ &= \sum_{k=0}^{9} \left| \frac{\Gamma(10 - k)}{\Gamma(\frac{3}{4})\Gamma(10 - k + \frac{1}{4})} \right| \\ &\approx 5.4134, \end{aligned}$$

that is, $M = 5.4134$. On the other hand,

$$\begin{aligned} |f(t, v(t)| &= \left| \frac{te^{-t}}{40} \sin v(t) \right| \\ &= \frac{te^{-t}}{40} |\sin v(t)|, \end{aligned}$$

Since $te^{-t} \leq \frac{1}{e}$ for $t > 0$, it follows that

$$|f(t, v(t)| \leq \frac{1}{40e} |v(t)|,$$

that is, $L = \frac{1}{40e}$, which implies $0 < 2ML(b - t_0) \approx 0.8964 < 1$. Then the conditions of Theorem 4.3 hold and the initial value problem has a unique solution.

Table 1: The values of $u_j(t)$, $i = 0, 1, 2, 3$ with $u_0(t) = \frac{t-1}{10t}$ on $[1, 10]_{\mathbb{T}}$.

t_i	$u_0(t_i)$	$u_1(t_i)$	$u_2(t_i)$	$u_3(t_i)$
1	0.00000000	0.00000000	0.00000000	0.00000000
2	0.05000000	0.00000000	0.00000000	0.00000000
3	0.06666667	0.00030448	0.00000000	0.00000000
4	0.07500000	0.00046754	0.00000102	0.00000000
5	0.08000000	0.00051924	0.00000159	0.00000000
6	0.08333333	0.00051857	0.00000174	0.00000000
7	0.08571429	0.00049932	0.00000171	0.00000000
8	0.08750000	0.00047630	0.00000163	0.00000000
9	0.08888889	0.00045480	0.00000155	0.00000000
10	0.09000000	0.00043617	0.00000147	0.00000000

To observe the behaviour of the Picard sequence of successive approximations, we computed the first 3 terms of the sequence for two different initial terms $u_0(t)$.

First, we took $u_0(t) = \frac{t-1}{10t}$ and computed $u_j(t)$, for $j = 1, 2, 3$ using

$$u_j(t) = Su_{j-1}(t) = \int_1^t \frac{1}{40} h_{-\frac{1}{4}}(t, \tau + 1)\tau e^{-\tau} \sin(u_{j-1}(\tau))\Delta\tau.$$

Figure 1 shows the the graphs of $u_j(t)$, for $j = 0, 1, 2, 3$ for $u_0(t) = \frac{t-1}{10t}$. In Table 1, we listed the computed values of $u_j(t)$, for $j = 0, 1, 2, 3$ for $u_0(t) = \frac{t-1}{10t}$.

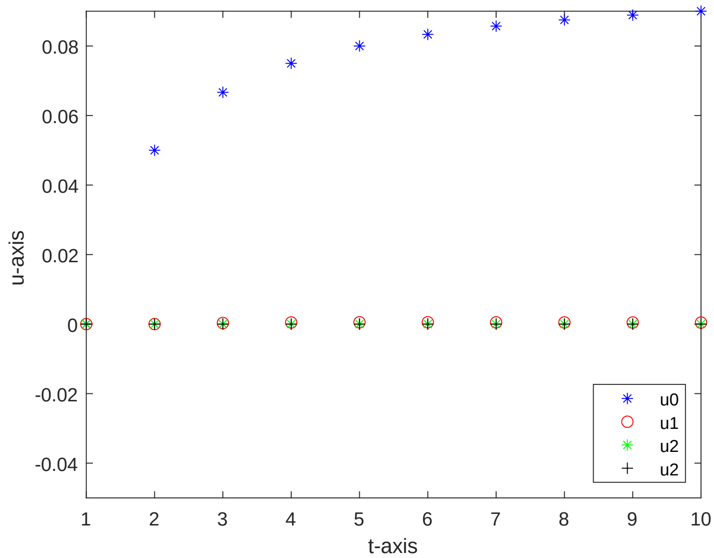

Figure 1: The graphs of $u_j(t)$, $i = 0, 1, 2, 3$ for $u_0(t) = \frac{t-1}{10t}$.

Our second choice for an initial term of the Picard sequence is $u_0(t) = \frac{1-t^2}{10t^3}$, which takes negative values on the interval $[1, 10]_{\mathbb{T}}$. We computed $u_j(t)$, for $j = 1, 2, 3$ from the same relation as above. Figure 2 shows the the graphs of $u_j(t)$, for $j = 0, 1, 2, 3$ for $u_0(t) = \frac{1-t^2}{10t^3}$. In Table 2, we listed the computed values of $u_j(t)$, for $j = 0, 1, 2, 3$ for $u_0(t) = \frac{1-t^2}{10t^3}$.

As a second example, we consider an initial value problem on the time scale of the form $\mathbb{T} = a\mathbb{N}_0$ which is used in applications of problems defined on evenly spaced discrete sets with arbitrary spacings.

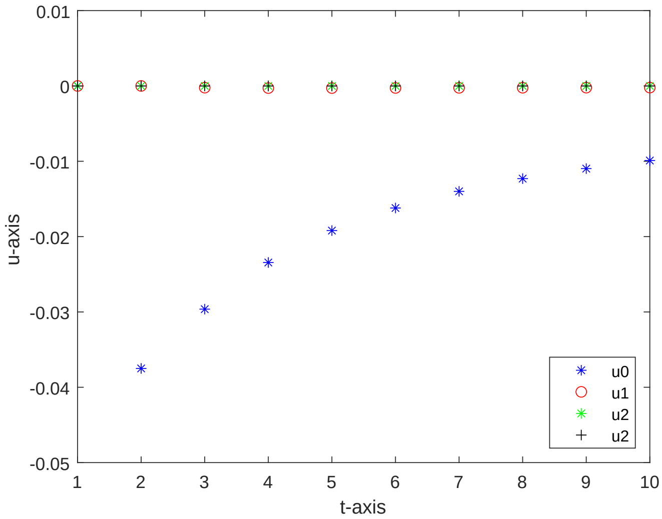

Figure 2: The graphs of $u_j(t)$, $i = 0, 1, 2, 3$ for $u_0(t) = \frac{1-t^2}{10t^3}$.

Table 2: The values of $u_j(t)$, $i = 0, 1, 2, 3$ with $u_0(t) = \frac{1-t^2}{10t^3}$ on $[1, 10]_{\mathbb{T}}$.

t_i	$u_0(t_i)$	$u_1(t_i)$	$u_2(t_i)$	$u_3(t_i)$
1	0.00000000	0.00000000	0.00000000	0.00000000
2	-0.03750000	0.00000000	0.00000000	0.00000000
3	-0.02962963	-0.00022841	0.00000000	0.00000000
4	-0.02343750	-0.00028232	-0.00000077	0.00000000
5	-0.01920000	-0.00028074	-0.00000108	0.00000000
6	-0.01620370	-0.00026622	-0.00000113	0.00000000
7	-0.01399417	-0.00025103	-0.00000109	0.00000000
8	-0.01230469	-0.00023799	-0.00000104	0.00000000
9	-0.01097394	-0.00022725	-0.00000099	0.00000000
10	-0.00990000	-0.00021841	-0.00000094	0.00000000

Example 4.5. *Let* $\mathbb{T} = \frac{1}{4}\mathbb{N}_0$ *and* $T = [t_0, b] = [0, 5]_{\mathbb{T}}$*. Consider the initial value problem*

$$\begin{cases} {}^C D^{11/2}_{\Delta,0} u(t) &= \dfrac{1}{400(1+t^2)} \ln(1 + |u(t)|), \quad t \in T, \\ u(0) &= 0, u^{\Delta}(0) = 0, \dots, u^{\Delta^{(4)}} = 0. \end{cases} \tag{15}$$

Here we have $\alpha = \frac{11}{2}$ *and* $m = -\overline{[-\frac{11}{2}]} = 5$*. On the time scale the fractional generalized* Δ*-power function is given as [13].*

$$h_\alpha(t, s) = \frac{\Gamma_a(t - s + a\alpha)}{\Gamma(\alpha + 1)\Gamma_a(t - s))}$$

where $a = \frac{1}{4}$*. The function* Γ_a *for time scale* $a\mathbb{N}_0$ *is given as by [10],*

$$\Gamma_a(x) = \int_0^\infty t^{x-1} e^{-(t^a/a)} dt$$

and one can easily see that it is related to the classical Gamma function by

$$\Gamma_a(x) = a^{\frac{x}{a}-1}\Gamma(x/a).$$

Hence, the generalized Δ*-power function on* $a\mathbb{N}_0$ *becomes*

$$h_\alpha(t,s) = \frac{a^{\frac{t-s+a\alpha}{a}-1}\Gamma(\frac{t-s+a\alpha}{a})}{\Gamma(\alpha+1)a^{\frac{t-s}{a}-1}\Gamma(\frac{t-s}{a})} = \frac{a^\alpha\Gamma(\frac{t-s}{a}+\alpha)}{\Gamma(\alpha+1)\Gamma(\frac{t-s}{a})}$$

so that we have

$$h_{9/2}(t,s) = \frac{(1/4)^{\frac{9}{2}}\Gamma(4(t-s)+\frac{9}{2})}{\Gamma(11/2)\Gamma(4(t-s)}.$$

Then, using the fact that on the time scale $a\mathbb{N}_0$*, we have* $\mu(t)=a$*,* $\sigma(t)=t+a$ *and*

$$\int_c^d g(y)\Delta y = \sum_{k=c/a}^{d/a-1} g(k)\mu(k) = a\sum_{k=c/a}^{d/a-1} g(k),$$

we compute

$$\begin{aligned}\int_0^t |h_{9/2}(t,\sigma(\tau))|\Delta\tau &= \int_0^t |h_{9/2}(t,\tau+1/4)|\Delta\tau \\ &\le \int_0^5 |h_{9/2}(t,\tau+1/4)|\Delta\tau \\ &= \frac{1}{4}\sum_{k=0}^{19}\left|\frac{(1/4)^{\frac{9}{2}}\Gamma(20-k+\frac{7}{2})}{\Gamma(11/2)\Gamma(20-k-1)}\right| \\ &= \frac{(1/4)^{\frac{11}{2}}}{\Gamma(\frac{11}{2})}\sum_{k=0}^{19}\left|\frac{\Gamma(20-k+\frac{7}{2})}{\Gamma(20-k-1)}\right| \\ &\approx 33.37,\end{aligned}$$

that is, $M=33.37$*. Also,*

$$\begin{aligned}|f(t,v(t)| &= \frac{1}{400(1+t^2)}|\ln(1+|v(t)|)| \\ &\le \frac{1}{400}|v(t)|,\end{aligned}$$

due to the fact that $\ln(1+|x|)\le|x|$*. Then* $L=\frac{1}{400}$ *and hence,* $0<2ML(b-t_0)=\frac{334}{400}=0.835<1$*. Then the conditions of Theorem 4.3 hold and the problem has a unique solution. To observe the behaviour of the Picard sequence of successive approximations, we computed the first 3 terms of the sequence for the initial terms* $u_0(t)=\frac{t^5}{20(1+t^3}$*.*

We computed $u_j(t)$*, for* $j=1,2,3$ *using*

$$u_j(t) = Su_{j-1}(t) = \int_1^t \frac{1}{400}h_{\frac{9}{4}}(t,\tau+\frac{1}{4})\frac{1}{1+\tau^2}\ln(1+|u_{j-1}(\tau)|)\Delta\tau.$$

Figure 3 shows the the graphs of $u_j(t)$*, for* $j=0,1,2,3$ *for* $u_0(t)=\frac{t^5}{20(1+t^3)}$*. In Table 3, we listed the computed values of* $u_j(t)$*, for* $j=0,1,2,3$ *for* $u_0(t)=\frac{t^5}{20(1+t^3)}$*.*

5. Conclusion

In this paper, an initial value problem with homogeneous initial conditions for a fractional dynamic equation on time scale is studied in the framework of metric-like space. Conditions for existence and uniqueness of the solution given in the main theorem are very easy to check, in comparison with the conditions given in other theorems existing in the literature. The results are supported by examples and numerical calculations.

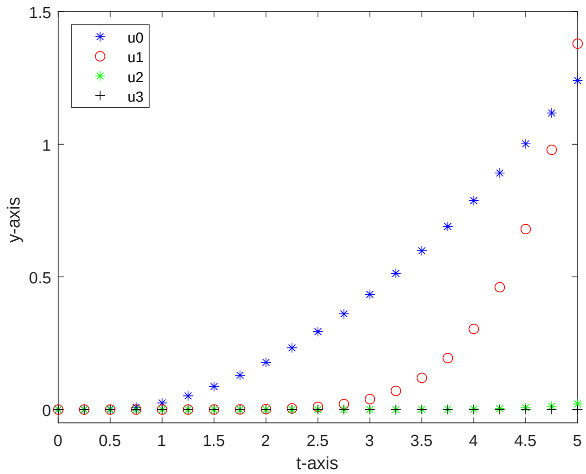

Figure 3: The graphs of $u_j(t)$, $i = 0, 1, 2, 3$ for $u_0(t) = \frac{t^5}{20(1+t^3)}$.

Table 3: The values of $u_j(t)$, $i = 0, 1, 2, 3$ with $u_0(t) = \frac{t^5}{20(1+t^3)}$ on $[0, 5]_{\mathbb{T}}$.

t_i	$u_0(t_i)$	$u_1(t_i)$	$u_2(t_i)$	$u_3(t_i)$
0.00	0.00000000	0.00000000	0.00000000	.00000000
0.25	0.00004808	0.00000000	0.00000000	0.00000000
0.50	0.00138889	0.00000000	0.00000000	0.00000000
0.75	0.00834478	0.00000011	0.00000000	0.00000000
1.00	0.02500000	0.00000340	0.00000000	0.00000000
1.25	0.05166997	0.00003059	0.00000000	0.00000000
1.50	0.08678571	0.00015867	0.00000001	0.00000000
1.75	0.12904638	0.00059137	0.00000006	0.00000000
2.00	0.17777778	0.00176426	0.00000037	0.00000000
2.25	0.23269625	0.00448385	0.00000178	0.00000000
2.50	0.29370301	0.01009854	0.00000683	0.00000000
2.75	0.36077733	0.02069837	0.00002231	0.00000000
3.00	0.43392857	0.03934097	0.00006403	0.00000001
3.25	0.51317586	0.07030064	0.00016574	0.00000004
3.50	0.59853989	0.11933820	0.00039393	0.00000015
3.75	0.69003980	0.19398935	0.00087156	0.00000046
4.00	0.78769231	0.30386917	0.00181382	0.00000128
4.25	0.89151158	0.46099137	0.00357995	0.00000331
4.50	1.00150950	0.68010015	0.00674506	0.00000801
4.75	1.11769600	0.97901358	0.01219615	0.00001830
5.00	1.24007937	1.37897698	0.02125590	0.00003978

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