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Numerical Analysis of Discrete Laplace Transform for Higher Order Cosine Function and Applications in Initial Value Problem

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Abstract. In this research work, we obtain Discrete Laplace Transform(DILAT) of higher order cosine functions and using this DILAT results by employing Initial Value Problems to get applications in the field of physical sciences. Also we present several theorems and examples to illustrate our findings by using MATLAB.

Keywords: Difference equation, Difference operator, Laplace transform, Initial value problem.

AMS classification: 39A70, 39A10, 47B39, 80A20

1. Introduction

The Laplace transform is a mathematical technique used to transform functions from the time domain to the frequency domain. It is a widely used tool in various fields, including mathematics, physics, engineering, and control systems. The Laplace transform is named after Pierre-Simon Laplace, a French mathematician and astronomer who introduced the transform in the late 18^{th} century. It provides a powerful method for solving differential equations and analyzing linear time-invariant systems[7]. When applying the Laplace transform, a function of time is converted into a function of a complex variable, usually denoted as "s". The transformed function, known as the Laplace transform of the original function, can provide valuable insights into the behavior of the system. The Laplace transform is defined as an integral, given by:

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$F(s) = L[f(t)] = \int_0^{\infty} e^{-st} f(t)dt = \sum_0^{\infty} e^{-st} f(t)$, where $F(s)$ represents the Laplace transform of $f(t)$, and s is a complex variable. The function $f(t)$ is multiplied by the exponential term e^{-st} and integrated over the interval $[0, \infty]$. By applying the Laplace transform, various operations such as differentiation, integration, and convolution can be simplified into algebraic operations in the frequency domain. This makes it easier to solve differential equations, analyze system responses, and study system stability and control. It is important to note that the Laplace transform is a well-established mathematical concept and is widely taught and used in academic and professional settings[11]. The Laplace transform allows us to separate the transient and steady-state components of a solution to an IVP[12, 13]. By transforming the given initial conditions and differential equation, we can solve for the Laplace transform of the solution. Inverse Laplace transforms the resulting which expression yields the solutions in the time domain, revealing the transient behavior (initial response) and the steady-state behavior (long-term response) of the system. At recently, the authors [8, 9, 10], have been obtained and analysed with many applications using Laplace transform with $n-$tuples.

The discrete Laplace transform is commonly used in digital signal processing, control systems analysis, and communication systems. It allows us to analyze the behavior of discrete systems and solve difference equations in the frequency domain[12]. Given a discrete-time signal or sequence, denoted as $x[n]$, the discrete Laplace transform, denoted as $X(z)$, is defined as: $X(z) = Z\{x[n]\} = \sum_0^{\infty} x[n]z^{-n}$, where z is a complex variable. The sequence $x[n]$ is multiplied by the term z^{-n} and summed over the range of n from 0 to infinity.

An initial value problem (IVP) is a type of differential equation problem that involves finding a solution to a differential equation, given an initial condition [4]. It typically consists of a differential equation and one or more initial conditions that specify the values of the unknown function(s) at a particular point. Mathematically, an initial value problem can be represented as follows: $\frac{dy}{dx} = f(x, y)y(x_0) = y_0$. Here, dy/dx represents the derivative of the unknown function y with respect to x, $f(x, y)$ is a given function that defines the relationship between x, y and their derivatives, x_0 is the initial value of x, and y_0 is the initial value of y at x_0. The goal is to find a function $y(x)$ that satisfies the differential equation with also satisfies the initial condition $y(x_0) = y_0$. The solution to the initial value problem is typically an expression for $y(x)$ that depends on the given differential equation and initial conditions. There are various methods to solve initial value problems depending on the type of differential equation involved. Some common techniques include separation of variables, integrating factors, variation of parameters, and numerical methods such as Euler's method or the Runge-Kutta method. Solving an initial value problem allows us to understand how the unknown function evolves over the domain of interest, given its initial behavior. These problems have applications in many fields, including physics, engineering, economics, and biology, where the behavior of systems can be modeled using differential equations. This research aims to propose a novel approach to solving initial value problems using DILAT.

2. Preliminaries

In this section, we present some basic definitions and results.

Lemma 2.1. [4] *For $k \in [a, b]$ and if $\ell = \dfrac{b-a}{M}$, then we have*

$$\Delta_\ell^{-1} u(k)\Big|_b^a = \sum_{r=1}^{M} u(b - r\ell) = \sum_{r=0}^{M-1} u(a + r\ell).$$

In general, we express

$$\Delta_\ell^{-1} u(k)\Big|_k^j = \sum_{r=1}^{[\frac{k}{\ell}]} u(k - r\ell) = \sum_{r=0}^{[\frac{k}{\ell}]-1} u(j + r\ell), k \in (\ell, \infty), j = k - [\frac{k}{\ell}]\ell.$$

Definition 2.2. [5] Let $u(k)$ and $v(k)$ be a complex valued functions defined on [a,b] and $\ell = \frac{b-a}{M}$. The discrete inner product of u and v with respect to ℓ is defined as

$$(u, v)_\ell = \ell \Delta_\ell^{-1} u(k) v^*(k)\Big|_a^b = \ell \sum_{r=0}^{M-1} u(a + r\ell) v^*(a + r\ell)$$

and the norm of u related to ℓ is defined as

$$\|u\|_{(\ell)} = (u, u)_\ell^{\frac{1}{2}} = \left[\ell \left(\Delta_\ell^{-1} |u(k)|^2\right)\Big|_a^b\right]^{\frac{1}{2}} = \left[\ell \sum_{r=0}^{M-1} \left|u(a + r\ell)^2\right|\right]^{\frac{1}{2}}$$

Definition 2.3. [11] Let $u(k)$, $k \in [\ell, \infty)$, be a real or complex valued function. Then, the ℓ-difference operator Δ_ℓ on $u(k)$ is defined as $\Delta_\ell u(k) = u(k + \ell) - u(k)$ and its infinte ℓ-difference sum is defined by $\Delta_\ell^{-1} u(k) = \sum\limits_{r=0}^{\infty} u(k + r\ell)$.

Definition 2.4. [6] Let $u(k)$ and $v(k)$ are the two real valued functions defined on $(-\infty, \infty)$ and if $\Delta_\ell v(k) = u(k)$, then the finite inverse principle law is given by

$$v(k) - v(k - \beta\ell) = \sum_{r=1}^{\beta} u(k - r\ell), \beta \in Z^+.$$

Applying the definition 2.2.1, we get the modified identities as follows:
(i) $\Delta_\ell k_\ell^{(\beta)} = (\beta\ell) k_\ell^{(\beta-1)}$, (ii) $\Delta_\ell^{-1} k_\ell^{(\beta)} = \frac{k_\ell^{(\beta+1)}}{\ell(\beta+1)}$, (iii) $\Delta_\ell^{-1} k^\beta = \sum\limits_{r=1}^{\beta} \frac{S_r^\beta \ell^{\beta-r} k_\ell^{(r+1)}}{(r+1)\ell}$.

Lemma 2.5. *Let $\ell > 0$ and $u(k), v(k)$ are real valued bounded functions. Then*

$$\Delta_\ell^{-1}(u(k)v(k)) = u(k)\Delta_\ell^{-1} v(k) - \Delta_\ell^{-1}\left(\Delta_\ell^{-1} v(k+\ell)\Delta_\ell u(k)\right).$$

Theorem 2.6. *If a function $f(x, y)$ is continuous in a region of the (x, y) plane, then for any initial condition (x_0, y_0), there exists a unique solution to the initial value problem $\frac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$ in some interval containing x_0.*

Lemma 2.7. *If there exists a constant L such that $|f(x, y_1) - f(x, y_2)| \leq L|y_1 - y_2|$ for all (x, y_1) and (x, y_2) in a region, then the solution to the initial value problem is unique in that region.*

Theorem 2.8. *If the function $f(x, y)$ is continuous and satisfies a Lipschitz condition with respect to y in a region, then there exists a unique solution to the initial value problem $d\frac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$.*

Definition 2.9. The discrete Laplace transform(DILAT) of a function $f(x)$, defined for all real numbers $x \geq 0$, is the function $F(s)$, defined by

$$F(s) = DILAT\{f(x)\} = \sum_{0}^{\infty} e^{-sx} f(x). \tag{1}$$

3. Main Results

Theorem 3.1. *If $x \in R$, then the discrete laplace transform of $\cos nx$ is*

$$DILAT(\cos nx) = \frac{e^{2s} - e^s \cos n}{e^{2s} - 2e^s \cos n + 1}.$$

Proof. $L(\cos nx) = \sum\limits_{x=0}^{\infty} \cos nx \; e^{-sx} = \sum\limits_{x=0}^{\infty} e^{nix} e^{-sx} = \sum\limits_{x=0}^{\infty} e^{x(ni-s)}$

$$= 1 + e^{in-s} + e^{2(in-s)} + \ldots = (1 - e^{(in-s)})^{-1} = \frac{1}{1 - e^{(ni-s)}}$$

$$= \frac{1}{1-e^{in}.e^{-s}} = \frac{e^s}{e^s - e^{in}} \quad = \frac{e^s}{e^s - \cos n - i\sin n}$$

$$= \frac{e^s}{e^s - \cos n - i\sin n} \times \frac{e^s - \cos n + i\sin n}{e^s - \cos n + i\sin n}$$

$$= \frac{e^{2s} - e^s\cos n + ie^s\sin n}{e^{2s} - 2e^s\cos n + \cos^2 n + \sin^2 n}$$

$$L(\cos nx) = \frac{e^{2s} - e^s\cos n}{e^{2s} - 2e^s\cos n + (\cos^2 n + \sin^2 n)}$$

Which completes the proof of Theorem 3.1. □

Theorem 3.2. *If $x \in R$, then the DILAT of* $\sin nx$ *is*

$$L(\sin nx) = \frac{e^s\sin n}{e^{2s} - 2e^s\cos n + 1}.$$

Proof. $L(\sin nx) = \sum_{x=0}^{\infty} e^{nix}\ e^{-xs} = \sum_{x=0}^{\infty} e^{x(ni-s)} = 1 + e^{ni-s} + e^{2(ni-s)} + \ldots$

$$= (1-e^{(ni-s)})^{-1} \quad = \frac{1}{1-e^{(ni-s)}} = \frac{e^s}{e^s - e^n i} \quad = \frac{e^s}{e^s - \cos n - i\sin n}$$

$$= \frac{e^{2s} - e^s\cos n + ie^s\sin n}{e^{2s} - 2e^s\cos n + \cos^2 n + \sin^2 n} = \frac{e^s\sin n}{e^{2s} - 2e^s\cos n + (\cos^2 n + \sin^2 n)}$$

From this we get the proof of Theorem 3.2. □

Theorem 3.3. *If $x \in R$, then the DILAT of $cos^2 x$ is*

$$L(\cos^2 x) = \frac{1}{2}\left[\frac{e^s}{e^s - 1} + \frac{e^{2s} - e^s\cos 2}{e^{2s} - 2e^s\cos 2 + 1}\right].$$

Proof. We know that $L(\cos^2 x) = L\left[\frac{1}{2}(1+\cos 2x)\right] = \frac{1}{2}\left[L[1] + L(\cos 2x)\right]$

By applying Theorem 3.1 we get the proof. □

Theorem 3.4. *If $x \in R$, then the DILAT of* $\cos^3 x$ *is*

$$\frac{1}{2^2}\left[\frac{e^{2s} - e^s cos3}{e^{2s} - 2e^s cos3 + 1} + 3\left(\frac{e^{2s} - e^s cos1}{e^{2s} - 2e^s cos1 + 1}\right)\right]$$

Proof. We have $L(\cos^3 x) = \frac{1}{2^2}L\left[\cos 3x + 3\cos x\right] = \frac{1}{2^2}\left[L(\cos 3x) + 3L(\cos x)\right]$

By applying Theorem (3.1) we get the proof. □

Theorem 3.5. *If $x \in R$, then the DILAT of* $\cos^4 x$ *is*

$$\frac{1}{2^3}\left[\frac{e^{2s}-e^s cos4}{e^{2s}-2e^s cos4+1}+4\left(\frac{e^{2s}-e^s cos2}{e^{2s}-2e^s cos2+1}\right)+\frac{3e^s}{e^s-1}\right].$$

Proof. Consider

$$L(\cos^4 x)=\frac{1}{2^3}L\left[\cos 4x+4\cos 2x+3\right]=\frac{1}{2^3}\left[L(\cos 4x)+4L(\cos 2x)+L(3)\right]$$

By applying Theorem (3.1) we get the proof. □

Theorem 3.6. *If $x \in R$, then the DILAT of* $\cos^5 x$ *is*

$$\frac{1}{2^4}\left[\frac{e^{2s}-e^s cos5}{e^{2s}-2e^s cos5+1}+5\left(\frac{e^{2s}-e^s cos3}{e^{2s}-2e^s cos3+1}\right)+10\left(\frac{e^{2s}-e^s cos1}{e^{2s}-2e^s cos1+1}\right)\right].$$

Proof. From the trigonometric expansion of $\cos^5 x$ and Theorem (3.1) we get the proof. □

Theorem 3.7. *If $x \in R$, then the DILAT of* $\cos^6 x$ *is*

$$\frac{1}{2^5}\left[\frac{e^{2s}-e^s cos6}{e^{2s}-2e^s cos6+1}+6\left(\frac{e^{2s}-e^s cos4}{e^{2s}-2e^s cos4+1}\right)+15\left(\frac{e^{2s}-e^s cos2}{e^{2s}-2e^s cos2+1}\right)+\frac{10e^s}{e^s-1}\right].$$

Proof. We get the proof by applying the expansion of $\cos^6 x$ and Theorem (3.1). □

Theorem 3.8. *If $x \in R$, then the DILAT of* $\cos^n x$ *is*

(i) If n is odd, then

$$L(cos^n x)=\frac{1}{2^{n-1}}\sum_{r=0}^{[\frac{n}{2}]} nc_r\frac{e^{2s}-e^s cos(n-2r)}{e^{2s}-2e^s cos(n-2r)+1} \tag{2}$$

(ii) If n is even, then

$$L(cos^n x)=\frac{1}{2^{n-1}}\sum_{r=0}^{[\frac{n-1}{2}]} nc_r\frac{e^{2s}-e^s cos(n-2r)}{e^{2s}-2e^s cos(n-2r)+1}+\frac{1}{2^n}(nc_{\frac{n}{2}}) \tag{3}$$

Proof. The proof of equation (2) obtain by using Theorem3.4 and Theorem 3.6 and proof of the equation (3) obtain by using Theorem 3.3, 3.5 and 3.7 for the induction on n. □

Example 3.9. *Taking $n = 15$ in equation (2), we get DILAT of* $\cos^{15} x$ *is*

$$L(\cos^{15} x)=\frac{1}{2^{14}}\left[\frac{e^{2s}-e^s\cos 15}{e^{2s}-2e^s\cos 15+1}+15\left(\frac{e^{2s}-e^s\cos 13}{e^{2s}-2e^s\cos 13+1}\right)\right.$$

$$+105\left(\frac{e^{2s}-e^{s}\cos 11}{e^{2s}-2e^{s}\cos 11+1}\right)+455\left(\frac{e^{2s}-e^{s}\cos 9}{e^{2s}-2e^{s}\cos 9+1}\right)$$
$$+1365\left(\frac{e^{2s}-e^{s}\cos 7}{e^{2s}-2e^{s}\cos 7+1}\right)+3003\left(\frac{e^{2s}-e^{s}\cos 5}{e^{2s}-2e^{s}\cos 5+1}\right)$$
$$+5005\left(\frac{e^{2s}-e^{s}\cos 3}{e^{2s}-2e^{s}\cos 3+1}\right)+6435\left(\frac{e^{2s}-e^{s}\cos 1}{e^{2s}-2e^{s}\cos 1+1}\right)\Bigg].$$

Example 3.10. *Taking $n = 18$ in equation (3), we get DILAT of $\cos^{18} x$ is*

$$L(\cos^{18} x)=\frac{1}{2^{17}}\Bigg[\frac{e^{2s}-e^{s}\cos 18}{e^{2s}-2e^{s}\cos 18+1}+18\left(\frac{e^{2s}-e^{s}\cos 16}{e^{2s}-2e^{s}\cos 16+1}\right)$$
$$+153\left(\frac{e^{2s}-e^{s}\cos 14}{e^{2s}-2e^{s}\cos 14+1}\right)+816\left(\frac{e^{2s}-e^{s}\cos 12}{e^{2s}-2e^{s}\cos 12+1}\right)$$
$$+3060\left(\frac{e^{2s}-e^{s}\cos 10}{e^{2s}-2e^{s}\cos 10+1}\right)+8568\left(\frac{e^{2s}-e^{s}\cos 8}{e^{2s}-2e^{s}\cos 8+1}\right)$$
$$+18564\left(\frac{e^{2s}-e^{s}\cos 6}{e^{2s}-2e^{s}\cos 6+1}\right)+31824\left(\frac{e^{2s}-e^{s}\cos 4}{e^{2s}-2e^{s}\cos 4+1}\right)$$
$$+43758\left(\frac{e^{2s}-e^{s}\cos 2}{e^{2s}-2e^{s}\cos 2+1}\right)+24310\Bigg].$$

4. IVP OF $\cos^n x$ USING DILAT

An Initial value problem (IVP)is an ordinary differential equation together with an initial condition which specifies the value of the unknown function at a given point in the domain. Modeling a system in physics or other sciences frequently amounts to solving an initial value problem. In this section we obtain solution of IVP By using DILAT.

Theorem 4.1. *The Discrete IVP $\Delta a_n = \cos x, a_0 = 1$ has a solution given by $a_n = 1 + cos1$.*

Proof. Let $\cos x = \Delta a_n \Leftrightarrow (e^s - 1)l(a_n) - a_0 = l(\Delta a_n)$

$$\Leftrightarrow (e^s-1)l(a_n)-1=l(\cos x)\Leftrightarrow l(a_n)=1+\frac{e^{2s}-e^{s}\cos 1}{(e^s-1)(e^{2s}-2e^{s}\cos 1+1)}$$

$$a_n = 1 + l^{-1}\left(\frac{e^{2s} - e^s\cos 1}{(e^s-1)(e^{2s} - 2e^s\cos 1 + 1)}\right)$$

$$a_n = 1 + l^{-1}\left(\frac{1}{e^s-1}\right) l^{-1}\left(\frac{e^{2s} - e^s\cos 1}{(e^{2s} - 2e^s\cos 1 + 1)}\right) = 1 + \cos 1 \qquad \square$$

Theorem 4.2. *The Discrete IVP* $\Delta a_n = \cos^2 x, a_0 = 1$ *has a solution given by* $a_n = 1 + \frac{1}{2}(\cos 2 + 1)$.

Proof. We have $\cos^2 x = \Delta a_n \Leftrightarrow (e^s-1)l(a_n) - a_0 = l(\Delta a_n)$

$$(e^s-1)l(a_n) - 1 = l(\cos^2 x) \Leftrightarrow l(a_n) = 1 + \frac{1}{2}\left[\frac{e^{2s} - e^s\cos 2}{(e^s-1)(e^{2s} - 2e^s\cos 2 + 1)} + \frac{1}{e^s-1}\right]$$

$$a_n = 1 + \frac{1}{2}\left[l^{-1}\left(\frac{e^{2s} - e^s\cos 2}{(e^s-1)(e^{2s} - 2e^s\cos 2 + 1)}\right) + l^{-1}\left(\frac{1}{e^s-1}\right)\right]$$

$$a_n = 1 + \frac{1}{2}\left[l^{-1}\left(\frac{1}{e^s-1}\right) l^{-1}\left(\frac{e^{2s} - e^s\cos 2}{(e^{2s} - 2e^s\cos 2 + 1)}\right) + l^{-1}\left(\frac{1}{e^s-1}\right)\right]$$

$$a_n = 1 + \frac{1}{2}(\cos 2 + 1) \qquad \square$$

Theorem 4.3. *The Discrete IVP* $\Delta a_n = \cos^3 x, a_0 = 1$ *has a solution given by* $a_n = 1 + \frac{1}{2^2}(cos3 + 3cos1)$.

Proof. Let $\cos^3 x = \Delta a_n \Leftrightarrow (e^s-1)l(a_n) - a_0 = l(\Delta a_n)$

$$(e^s-1)l(a_n) - 1 = l(\cos^3 x)$$

$$\Leftrightarrow (e^s-1)l(a_n) - 1 = \frac{1}{2^2}\left[\frac{e^{2s} - e^s\cos 3}{e^{2s} - 2e^s\cos 3 + 1} + 3\left(\frac{e^{2s} - e^s\cos 1}{e^s - 2e^s\cos 1 + 1}\right)\right]$$

$$l(a_n) = 1 + \frac{1}{2^2}\left[\frac{e^{2s} - e^s\cos 3}{(e^s-1)(e^{2s} - 2e^s\cos 3 + 1)} + 3\left(\frac{e^{2s} - e^s\cos 1}{(e^s-1)(e^s - 2e^s\cos 1 + 1)}\right)\right]$$

$$a_n = 1 + \frac{1}{2^2}\left[l^{-1}\left(\frac{e^{2s} - e^s\cos 3}{(e^s-1)(e^{2s} - 2e^s\cos 3 + 1)}\right) + 3l^{-1}\left(\frac{e^{2s} - e^s\cos 1}{(e^s-1)(e^s - 2e^s\cos 1 + 1)}\right)\right]$$

$$a_n = 1 + \frac{1}{2^2}(\cos 3 + 3\cos 1) \qquad \square$$

Theorem 4.4. *The Discrete IVP* $\Delta a_n = \cos^4 x, a_0 = 1$ *has a solution given by* $a_n = 1 + \frac{1}{2^3}(cos4 + 4(cos2) + 3)$.

Proof. We have $\cos^4 x = \Delta a_n \Leftrightarrow (e^s-1)l(a_n) - a_0 = l(\Delta a_n)$

$$\Leftrightarrow (e^s - 1)l(a_n) - 1 = l(\cos^4 x)$$

$$\Leftrightarrow (e^s - 1)l(a_n) - 1 = \frac{1}{2^3}\left[\frac{e^{2s} - e^s \cos 4}{e^{2s} - 2e^s \cos 4 + 1} + 4\left(\frac{e^{2s} - e^s \cos 2}{e^{2s} - 2e^s \cos 2 + 1}\right) + 3\right]$$

$$\Leftrightarrow (e^s - 1)l(a_n) = 1 + \frac{1}{2^3}\left[\frac{e^{2s} - e^s \cos 4}{e^{2s} - 2e^s \cos 4 + 1} + 4\left(\frac{e^{2s} - e^s \cos 2}{e^{2s} - 2e^s \cos 2 + 1}\right) + 3\right]$$

$$\Leftrightarrow l(a_n) = 1 + \frac{1}{2^3}\left[\frac{e^{2s} - e^s \cos 4}{(e^s - 1)(e^{2s} - 2e^s \cos 4 + 1)} + 4\left(\frac{e^{2s} - e^s \cos 2}{(e^s - 1)(e^{2s} - 2e^s \cos 2 + 1)}\right) + \frac{3}{e^s - 1}\right]$$

$$a_n = 1 + \frac{1}{2^3}\left(\cos 4 + 4(\cos 2) + 3\right)$$ □

Theorem 4.5. *The Discrete IVP $\Delta a_n = \cos^5 x, a_0 = 1$ has a solution given by $a_n = 1 + \frac{1}{2^4}\left(cos5 + 5(cos3) + 10(cos1)\right)$.*

Proof. We have $\cos^5 x = \Delta a_n \Leftrightarrow (e^s - 1)l(a_n) - a_0 = l(\Delta a_n)$

$$\Leftrightarrow (e^s - 1)l(a_n) - 1 = l(\cos^5 x)$$

$$\Leftrightarrow l(a_n) = 1 + \frac{1}{2^4}\left[\frac{e^{2s} - e^s \cos 5}{(e^s - 1)(e^{2s} - 2e^s \cos 5 + 1)} + 5\left(\frac{e^{2s} - e^s \cos 3}{(e^s - 1)(e^{2s} - 2e^s \cos 3 + 1)}\right)\right.$$

$$\Leftrightarrow a_n = 1 + \frac{1}{2^4}\left[l^{-1}\left(\frac{1}{e^s - 1}\right) l^{-1}\left(\frac{e^{2s} - e^s \cos 5}{(e^{2s} - 2e^s \cos 5 + 1)}\right) + 5l^{-1}\left(\frac{1}{e^s - 1}\right) l^{-1}\left(\frac{e^{2s} - e^s \cos 3}{(e^{2s} - 2e^s \cos 3 + 1)}\right) + 10l^{-1}\left(\frac{1}{e^s - 1}\right) l^{-1}\left(\frac{e^{2s} - e^s \cos 1}{(e^{2s} - 2e^s \cos 1 + 1)}\right)\right]$$

$$a_n = 1 + \frac{1}{2^4}\left(\cos 5 + 5(\cos 3) + 10(\cos 1)\right)$$ □

Theorem 4.6. *The Discrete IVP $\Delta a_n = \cos^6 x, a_0 = 1$ has a solution given by $a_n = 1 + \frac{1}{2^5}\left(cos6 + 6(cos4) + 15(cos2) + 10\right)$.*

Proof. We have $\cos^6 x = \Delta a_n \Leftrightarrow (e^s - 1)l(a_n) - a_0 = l(\Delta a_n)$

$$\Leftrightarrow (e^s - 1)l(a_n) - 1 = l(\cos^6 x)$$

$$\Leftrightarrow l(a_n) = 1 + \frac{1}{2^5}\left[\frac{e^{2s} - e^s cos6}{(e^s-1)(e^{2s} - 2e^s cos6 + 1)} + 6\left(\frac{e^{2s} - e^s cos4}{(e^s-1)(e^{2s} - 2e^s cos4 + 1)}\right)\right.$$
$$\left. +15\left(\frac{e^{2s} - e^s cos2}{(e^s-1)(e^{2s} - 2e^s cos2 + 1)}\right) + \frac{10}{e^s - 1}\right]$$

$$\Leftrightarrow a_n = 1 + \frac{1}{2^5}\left[l^{-1}\left(\frac{1}{e^S - 1}\right) l^{-1}\left(\frac{e^{2s} - e^s cos6}{(e^{2s} - 2e^s cos6 + 1)}\right)\right.$$
$$+6l^{-1}\left(\frac{1}{e^S - 1}\right) l^{-1}\left(\frac{e^{2s} - e^s cos4}{(e^{2s} - 2e^s cos4 + 1)}\right)$$
$$\left. +15l^{-1}\left(\frac{1}{e^S - 1}\right) l^{-1}\left(\frac{e^{2s} - e^s cos2}{(e^{2s} - 2e^s cos2 + 1)}\right) + l^{-1}\left(\frac{10}{e^s - 1}\right)\right]$$

$$\Leftrightarrow a_n = 1 + \frac{1}{2^5}\left(cos6 + 6(cos4) + 15(cos2) + 10\right)$$ □

Theorem 4.7. *The Discrete IVP $\Delta a_n = cos^k x, a_0 = 1$ has a solution given by*

$$a_n = 1 + \frac{1}{2^{n-1}}\sum_{r=0}^{[\frac{n}{2}]} nc_r cos(n-2r) \qquad n \text{ is odd}$$

$$a_n = 1 + \frac{1}{2^{n-1}}\sum_{r=0}^{[\frac{n-1}{2}]} nc_r cos(n-2r) + \frac{1}{2^n}(nc_{\frac{n}{2}}) \qquad n \text{ is even}$$

Proof. We shall prove this theorem by induction on k as two kind in odd and even cases.

For the odd particular values $k = 1, 3, 5$ we have the results in the Theorem (4.1),(4.3),(4.5) and for the even particular values $k = 2, 4, 6$ the results of Theorem (4.2),(4.4),(4.6) which we have proceeding like this induction on k, we get the proof. □

The following figures represents the numerical illustration of the cosine functions of power 1,2 and 3 and also the solution of initial value problems are presented for better understanding of our findings.

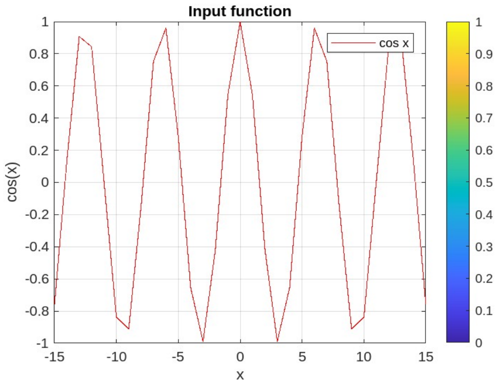

FIGURE 1. Input function $\cos x$

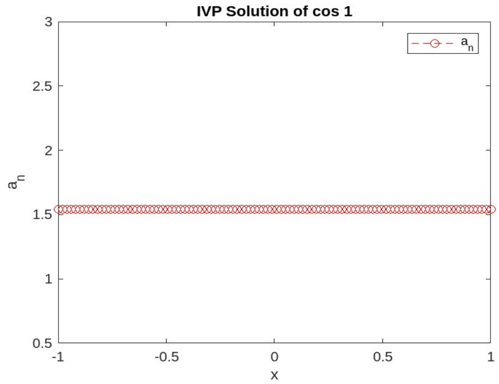

FIGURE 2. IVP solution of $\cos x$

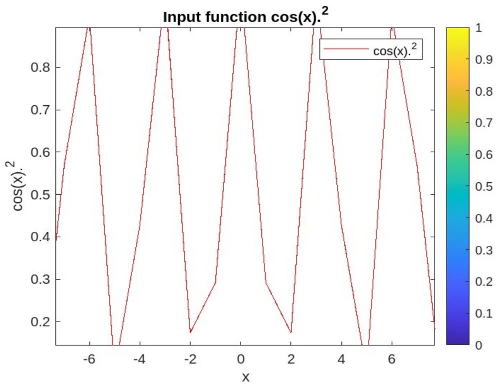

FIGURE 3. Input function $\cos^2 x$

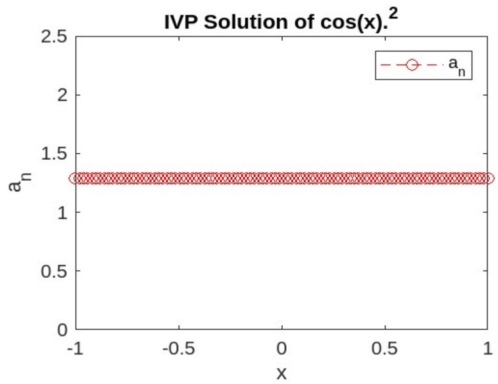

FIGURE 4. IVP solution of $\cos^2 x$

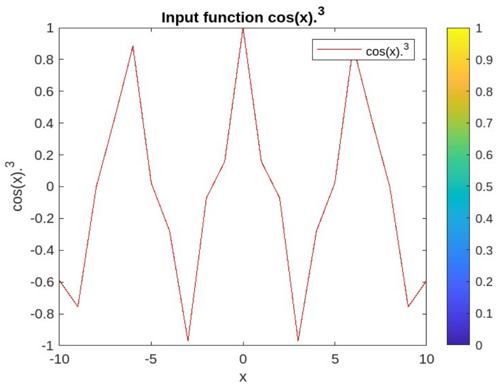

FIGURE 5. Input function $\cos^3 x$

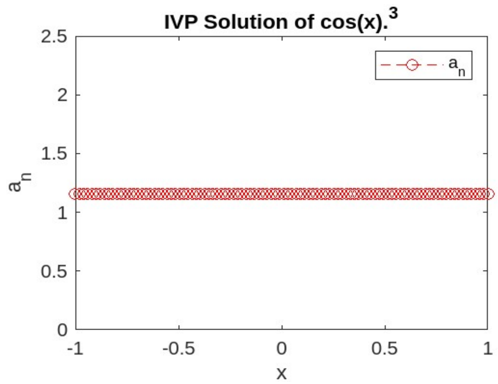

FIGURE 6. IVP solution of $\cos^3 x$

5. Conclusion

In this research work, the Discrete Laplace transform(DILAT) is successfully defined and analyzed with the help of difference operator. We have successfully derived formulas and results for the higher order cosine functions by applying initial value problems to obtain applications. Finally we conclude that this investigation describes the solutions of each initial value problems for our findings by using MATLAB in a graphical manner.

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