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On the dendrite property of fractal cubes

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Abstract

The paper considers the fractal cubes and presents all the data according to which it is possible to recognize whether a given fractal cube is a dendrite. The method of detecting the dendrite property for a fractal cube is based on the finding of the bipartite intersection graph for the fractal cube.

Keywords: fractal cube, dendrite, self-similar set, single intersection property, intersection graph
2010 MSC: 28A80

Let $n \geq 2$ and let $D = \{d_1, \cdots, d_N\} \subset \{0, 1, \ldots, n-1\}^k$. We call the set D a digit set. The set D and the integer n determine a system of contraction similarities $\mathcal{S} = \{S_j(x) = \frac{1}{n}(x + d_j)\}_{j=1}^N$ in $\mathbb{R}^k$, whose attractor K satisfies the set equation

$$K = \frac{K + D}{n} \tag{1}$$

The attractor K is called a *fractal k-cube* of order n with digit set D. In special cases where $k = 2$ and $k = 1$, K is called a *fractal square* and a *fractal segment*, respectively.

Though the topology of fractal squares was addressed by many authors [2, 5, 10, 3], the topology of fractal cubes is still waiting for the researchers attention. The purpose of the paper is to describe how to detect the sets D for which the fractal cube K is a dendrite. The authors were motivated by the question of Hui Rao: "How to detect a fractal square dendrite?" and proceed to find answer to this question and many related ones.

There is a sufficient condition for the dendrite property of self-similar sets [1, 7], which states that if a self-similar set K has the single intersection property and its bipartite intersection graph is a tree, then K is a dendrite. As proved in our previous paper [4], this condition is also necessary for fractal squares. We believe that this is true for fractal cubes as well.

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In Section 1, we present the results of our paper [8] that describe the system A_k of faces $P_{\boldsymbol{\alpha}}$ of the unit k-cube P^k and of a fractal k-cube K. In Section 2 we then formulate the intersection theorem for a pair K_1, K_2 of fractal cubes, which defines the system Σ of equations for the intersections $F_{\boldsymbol{\alpha}}$ of faces of these two fractal cubes.Then we consider the structural graph for that system.

In Section 3 we expose the results of our paper [7] on the single intersection property for self-similar sets and the intersection graph criterion for the dendrite property for self-similar sets, which initially appeared in [1]. In Section 4 we present a roadmap for defining whether the given fractal cube is a dendrite.

1. The faces of a fractal cube.

1.1. The system of faces of the unit cube P^k.

A fractal k-cube K is always a subset of the unit k-cube $P^k = [0,1]^k$ in $\mathbb{R}^k$.

The system of faces of the unit cube P^k is defined by the parameter set $A_k = \{-1, 0, 1\}^k$. Every vector $\boldsymbol{\alpha} = (\alpha_1, \ldots \alpha_k) \in A_k$ defines the unique face $P_{\boldsymbol{\alpha}}$ of the cube P^k through the equation $P^k \cap (P^k + \boldsymbol{\alpha}) = P_{\boldsymbol{\alpha}}$. The dimension $j_{\boldsymbol{\alpha}}$ of $P_{\boldsymbol{\alpha}}$ is equal to $k - |\boldsymbol{\alpha}|$, where $|\boldsymbol{\alpha}| = (|\alpha_1| + \ldots + |\alpha_k|)$, so $P_{\boldsymbol{\alpha}}$ is a unit $j_{\boldsymbol{\alpha}}$-cube.

There is an order relation $\sqsubseteq$ on the set A_k, defined by the rule:

$$\boldsymbol{\beta} \sqsubseteq \boldsymbol{\alpha} \textit{ if for any } i = 1, ..., k,\ \alpha_i \neq 0 \textit{ implies } \beta_i = \alpha_i.$$

Thus, $\boldsymbol{\beta} \sqsubseteq \boldsymbol{\alpha}$ iff $P_{\boldsymbol{\beta}}$ is a face of $j_{\boldsymbol{\alpha}}$-cube $P_{\boldsymbol{\alpha}}$. A vector $\boldsymbol{\alpha} \in A_k$ is maximal with respect to the relation $\sqsubset$, if and only if for any i, $\alpha_i \neq 0$. In this case $P_{\boldsymbol{\alpha}}$ is a vertex of the cube P^k.

We say that $\boldsymbol{\alpha}, \boldsymbol{\beta} \in A_k \setminus \{0\}$ are *complementary* and write $\boldsymbol{\alpha} \perp \boldsymbol{\beta}$, if $\sum\limits_{i=1}^{k} |\alpha_i \beta_i| = 0$. If $\boldsymbol{\alpha} \perp \boldsymbol{\beta}$, then $\boldsymbol{\alpha} + \boldsymbol{\beta} \in A_k$ and $\boldsymbol{\alpha} \sqsubset \boldsymbol{\alpha} + \boldsymbol{\beta}$. The relation $\boldsymbol{\alpha} \perp \boldsymbol{\beta}$ holds iff $P_{\boldsymbol{\alpha}} \cap P_{\boldsymbol{\beta}} = P_{\boldsymbol{\alpha}+\boldsymbol{\beta}}$. We denote by $A_{\boldsymbol{\alpha}}$ the set of all $\boldsymbol{\beta} \in A \setminus \{0\}$, complementary to $\boldsymbol{\alpha}$.

The set $\{\boldsymbol{\alpha} + P^k, \boldsymbol{\alpha} \in A \setminus \{0\}\}$ is the set of all neighbors of P^k in the family $\{d + P^k, d \in \mathbb{Z}^k\}$, and

$$\partial P^k = \bigcup_{\boldsymbol{\alpha} \in A_k \setminus \{0\}} P_{\boldsymbol{\alpha}} \tag{2}$$

Similarly, for each $\boldsymbol{\alpha} \in A$ the boundary of $P_{\boldsymbol{\alpha}}$ is represented by the equation

$$\partial P_{\boldsymbol{\alpha}} = \bigcup_{\boldsymbol{\beta} \sqsupset \boldsymbol{\alpha}} P_{\boldsymbol{\beta}} = \bigcup_{\gamma \in A_{\boldsymbol{\alpha}} \setminus \{0\}} P_{\boldsymbol{\alpha}+\gamma} \tag{3}$$

1.2. The faces of a fractal cube K.

Definition 1.1. *Let K be a fractal k-cube. For $\boldsymbol{\alpha} \in A_k \setminus \{0\}$, the set $K_{\boldsymbol{\alpha}} = K \cap P_{\boldsymbol{\alpha}}$ is called the $\boldsymbol{\alpha}$-face of K.*

Theorem 1.2. *For each $\boldsymbol{\alpha} \in A_k$, the set $K_{\boldsymbol{\alpha}} = K \cap P_{\boldsymbol{\alpha}}$ is a fractal k-cube with digit set $D_{\boldsymbol{\alpha}} = D \cap (n-1)P_{\boldsymbol{\alpha}}$.*

If $\boldsymbol{\alpha} \perp \boldsymbol{\beta}$ then $K_{\boldsymbol{\alpha}+\boldsymbol{\beta}} = K_{\boldsymbol{\alpha}} \cap K_{\boldsymbol{\beta}} = K \cap P_{\boldsymbol{\alpha}+\boldsymbol{\beta}}$ is a fractal k-cube with digit set $D_{\boldsymbol{\alpha}+\boldsymbol{\beta}} = D \cap (n-1)P_{\boldsymbol{\alpha}+\boldsymbol{\beta}}$. The formulas (2), (3) imply the following equalities.

$$\partial K = \bigcup_{\boldsymbol{\alpha} \in A \setminus \{0\}} K_{\boldsymbol{\alpha}}; \qquad \partial K_{\boldsymbol{\alpha}} = \bigcup_{\boldsymbol{\beta} \in A_{\boldsymbol{\alpha}} \setminus \{0\}} K_{\boldsymbol{\alpha}+\boldsymbol{\beta}}. \tag{4}$$

2. The equations for the intersection of fractal cubes

2.1. The digit sets G_α and $G_{\alpha\beta}$

Definition 2.1. *Let K_1, K_2 be fractal k-cubes of order n with digit sets D_1, D_2. Denote by $F_{\boldsymbol{\alpha}}$ the intersection sets $F_{\boldsymbol{\alpha}} = K_1 \cap (K_2 + \boldsymbol{\alpha})$, $\boldsymbol{\alpha} \in A_k$. In particular, $F_0 = K_1 \cap K_2$.*

To study the structure of $F_0 = K_1 \cap K_2$, one should take into account all intersection sets $F_{\boldsymbol{\alpha}}$ and establish the relations between these sets.

The $\boldsymbol{\alpha}$ and $(-\boldsymbol{\alpha})$ faces of the fractal cubes K_1 and K_2 are $K_{1,\boldsymbol{\alpha}} = K_1 \cap P_{\boldsymbol{\alpha}}$ and $K_{2,-\boldsymbol{\alpha}} = K_2 \cap P_{-\boldsymbol{\alpha}}$ respectively. Therefore, $F_{\boldsymbol{\alpha}} = K_{1,\boldsymbol{\alpha}} \cap (K_{2,-\boldsymbol{\alpha}} + \boldsymbol{\alpha})$.

According to Theorem 1.2, the sets on the right side of the equation may be considered as fractal cubes themselves; therefore, we have the following proposition.

Proposition 2.2. *Given fractal k-cubes K_1, K_2 of order n with digit sets D_1, D_2 and $\boldsymbol{\alpha} \in A_k$, the set $F_{\boldsymbol{\alpha}}$ is the intersection of fractal cubes $\hat{K}_1, \hat{K}_2$ with digit sets $\hat{D}_1 = D_1 \cap (n-1)P_{\boldsymbol{\alpha}}$ and $\hat{D}_2 = D_2 \cap (n-1)P_{-\boldsymbol{\alpha}} + (n-1)\boldsymbol{\alpha}$, respectively. Furthermore, for any $\boldsymbol{\gamma} \perp \boldsymbol{\alpha}$, $F_{\boldsymbol{\alpha}+\boldsymbol{\gamma}} = \hat{K}_1 \cap (\hat{K}_2 + \boldsymbol{\gamma})$.*

It follows from Proposition 2.2 that the intersection of fractal cubes $\hat{K}_1$ and $\hat{K}_2$ and all their faces may also be considered independently of the initial sets K_1, K_2.

Definition 2.3. *We denote by $G_{\boldsymbol{\alpha}}$ the intersection of the digit sets $\hat{D}_1 = D_1 \cap (n-1)P_{\boldsymbol{\alpha}}$ for $K_{1,\boldsymbol{\alpha}}$ and $\hat{D}_2 = D_2 \cap (n-1)P_{-\boldsymbol{\alpha}} + (n-1)\boldsymbol{\alpha}$ for $(K_{2,-\boldsymbol{\alpha}} + \boldsymbol{\alpha})$, which is equal to $D_1 \cap (D_2 + (n-1)\boldsymbol{\alpha})$. We denote by $G_{\boldsymbol{\alpha}\boldsymbol{\beta}}$ the set $D_1 \cap (D_2 + n\boldsymbol{\alpha} - \boldsymbol{\beta})$.*

The digit set $G_{\boldsymbol{\alpha}}$ is naturally associated with the set $F_{\boldsymbol{\alpha}}$. If $\boldsymbol{\alpha} = 0$, the set $G_{\boldsymbol{\alpha}}$ becomes $G_0 = D_1 \cap D_2$.

2.2. The intersection theorem and the graph Γ_Σ.

The following theorem establishes the relations between the sets $F_{\boldsymbol{\alpha}}$:

Theorem 2.4. *[8] The family $\{F_{\boldsymbol{\alpha}}, \boldsymbol{\alpha} \in A_k\}$ of intersections $F_{\boldsymbol{\alpha}} = K_1 \cap (K_2 + \boldsymbol{\alpha})$ satisfies the system Σ of equations.*

$$F_{\boldsymbol{\alpha}} = \bigcup_{\boldsymbol{\beta} \sqsupseteq \boldsymbol{\alpha}} T_{\boldsymbol{\alpha}\boldsymbol{\beta}}(F_{\boldsymbol{\beta}}), \qquad \boldsymbol{\alpha} \in A_k, \tag{5}$$

where for any $\boldsymbol{\beta} \sqsupseteq \boldsymbol{\alpha}$,

$$T_{\boldsymbol{\alpha}\boldsymbol{\beta}}(F_{\boldsymbol{\beta}}) = \frac{1}{n}(F_{\boldsymbol{\beta}} + G_{\boldsymbol{\alpha}\boldsymbol{\beta}}) \quad \textit{and} \qquad G_{\boldsymbol{\alpha}\boldsymbol{\beta}} = D_1 \cap (D_2 + n\boldsymbol{\alpha} - \boldsymbol{\beta}) \tag{6}$$

We consider the *structural graph* Γ_Σ of the system Σ defined by equations (5),(6), which is the main tool for finding the properties of sets $F_{\boldsymbol{\alpha}}$.

Definition 2.5. *The structural graph Γ_Σ is a directed graph, whose vertices are all $\boldsymbol{\alpha} \in A_k$, for which $F_{\boldsymbol{\alpha}} \neq \varnothing$. A directed edge in Γ_Σ from $\boldsymbol{\alpha}$ to $\boldsymbol{\beta}$ exists if and only if $\boldsymbol{\alpha} \sqsubseteq \boldsymbol{\beta}$ and the operator $T_{\boldsymbol{\alpha\beta}}$ is non-degenerate.*

In general, the graph Γ_Σ may contain 3^k vertices and 5^k edges, and 3^k of these edges are loops from $\boldsymbol{\alpha}$ to itself. We mark each edge with $G_{\boldsymbol{\alpha\beta}}$.

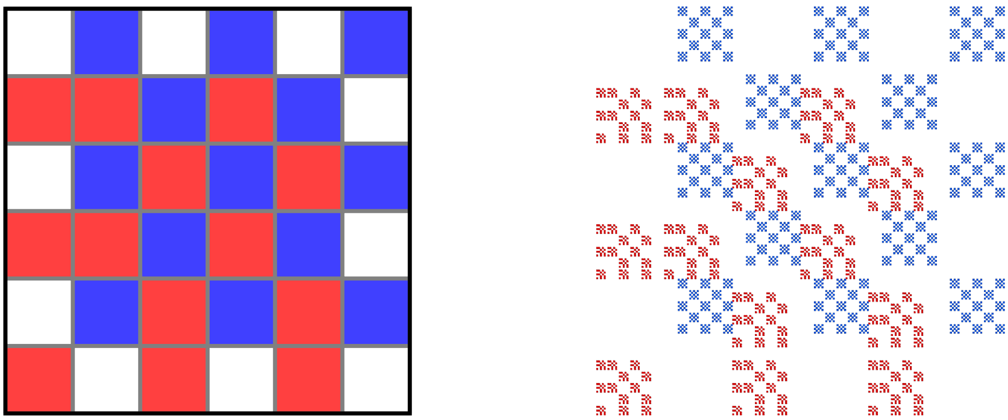

Figure 1: Intersection of two fractal squares

However, some of the vertices and edges in the graph Γ_Σ vanish. This occurs for $\boldsymbol{\alpha}$ such that $F_{\boldsymbol{\alpha}} = \varnothing$ and for those edges $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ for which $T_{\boldsymbol{\alpha\beta}}(F_{\boldsymbol{\beta}}) = \varnothing$.

It is obvious that

$$T_{\boldsymbol{\alpha\beta}}(F_{\boldsymbol{\beta}}) = \varnothing \quad \text{iff} \quad G_{\boldsymbol{\alpha\beta}} = \varnothing \quad \text{or} \quad F_{\boldsymbol{\beta}} = \varnothing \tag{7}$$

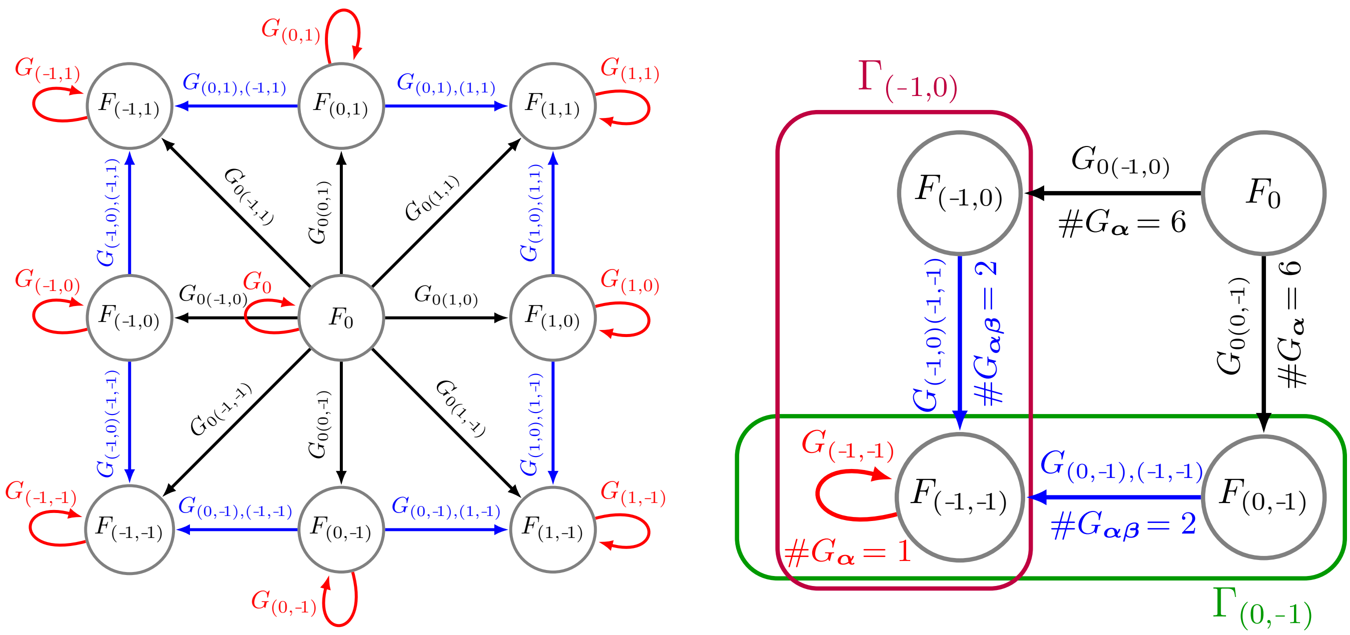

Figure 2: The maximal possible structure graph Γ_Σ for the intersection of two fractal squares (on the left) and the graph Γ_Σ for Example 1 (on the right). The picture on the right shows the subgraphs $\Gamma_{(-1,0)}$ and $\Gamma_{(0,-1)}$.

A set $F_{\boldsymbol{\alpha}}$ is empty if $G_{\boldsymbol{\alpha}} = \varnothing$ and for any $\boldsymbol{\beta} \sqsupset \boldsymbol{\alpha}$ the set $F_{\boldsymbol{\beta}} + G_{\boldsymbol{\alpha\beta}} = \varnothing$. Applying (7) to all $\boldsymbol{\beta} \sqsupset \boldsymbol{\alpha}$, we deduce the following emptiness condition for $F_{\boldsymbol{\alpha}}$:

Lemma 2.6. *[8] A set $F_{\boldsymbol{\alpha}} = \varnothing$ if and only if for any $\boldsymbol{\beta} \sqsupseteq \boldsymbol{\alpha}$ and for any finite sequence $\boldsymbol{\alpha} = \boldsymbol{\alpha}_0 \sqsubseteq \boldsymbol{\alpha}_1 \sqsubseteq \ldots \boldsymbol{\alpha}_{p-1} \sqsubseteq \boldsymbol{\alpha}_p = \boldsymbol{\beta}$,
the product $\#G_{\boldsymbol{\alpha}_0\boldsymbol{\alpha}_1}\#G_{\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2}\ldots\#G_{\boldsymbol{\alpha}_{p-1}\boldsymbol{\alpha}_p}\#G_{\boldsymbol{\beta}}$ is equal to zero.* □

For these reasons, due to the reduction of all empty vertices and empty edges, the structure graph Γ for the system Σ defined in Theorem 2.4 has the set of vertices $V_\Sigma = \{F_{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in A, F_{\boldsymbol{\alpha}} \neq \varnothing\}$ and the set of edges $E_\Sigma = \{(F_{\boldsymbol{\alpha}}, F_{\boldsymbol{\beta}}) : \boldsymbol{\alpha} \sqsubseteq \boldsymbol{\beta}, G_{\boldsymbol{\alpha\beta}} \neq \varnothing, F_{\boldsymbol{\beta}} \neq \varnothing\}$.

In general, the graph Γ_Σ may be disconnected.

We say that two vertices $F_{\boldsymbol{\alpha}}, F_{\boldsymbol{\beta}}, \boldsymbol{\alpha} \sqsubset \boldsymbol{\beta}$ *are connected by a directed path in* Γ_Σ, if there is a finite sequence $\boldsymbol{\alpha} = \boldsymbol{\alpha}_0 \sqsubset \boldsymbol{\alpha}_1 \sqsubset \ldots \boldsymbol{\alpha}_{p-1} \sqsubset \boldsymbol{\alpha}_p = \boldsymbol{\beta}$ such that for any $j = 0, \ldots, p$ sets $F_{\boldsymbol{\alpha}_j} \neq \varnothing$ and sets $G_{\boldsymbol{\alpha}_{j-1}\boldsymbol{\alpha}_j} \neq \varnothing$ for $j = 1, \ldots, p$.

We write $\boldsymbol{\beta} \succ \boldsymbol{\alpha}$ if there is a directed path in Γ from $F_{\boldsymbol{\alpha}}$ to $F_{\boldsymbol{\beta}}$.
If $\boldsymbol{\beta} \succcurlyeq \boldsymbol{\alpha}$ or $\boldsymbol{\alpha} \succcurlyeq \boldsymbol{\beta}$ then we say that $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are Γ-*comparable*.
We denote by $\Gamma_{\boldsymbol{\alpha}}$ a subgraph in Γ, whose vertices are all $F_{\boldsymbol{\beta}}$ such that $\boldsymbol{\beta} \succcurlyeq \boldsymbol{\alpha}$. The relation $\boldsymbol{\beta} \succcurlyeq \boldsymbol{\alpha}$ implies that $\Gamma_{\boldsymbol{\beta}}$ is contained in $\Gamma_{\boldsymbol{\alpha}}$.
We say that $\boldsymbol{\beta}$ is *maximal* for $\Gamma_{\boldsymbol{\alpha}}$, if $\Gamma_{\boldsymbol{\beta}}$ is a single vertex $F_{\boldsymbol{\beta}}$. We say that $\boldsymbol{\beta}$ is *minimal* for Γ_Σ, if there is no $\boldsymbol{\alpha}$ such that $\boldsymbol{\alpha} \prec \boldsymbol{\beta}$.

The following theorem states the conditions under which $F_{\boldsymbol{\alpha}}$ is countable, finite, or a single-point set.

Theorem 2.7. *[8] Let K_1, K_2 be fractal k-cubes of order n and Γ_Σ be the structural graph for the intersection of K_1 and K_2.
(a) If there is a vertex $\boldsymbol{\beta}$ in $\Gamma_{\boldsymbol{\alpha}}$, such that $\#G_{\boldsymbol{\beta}} > 1$, then the set $F_{\boldsymbol{\alpha}}$ is uncountable;
(b) If for all vertices $\boldsymbol{\beta}$ in $\Gamma_{\boldsymbol{\alpha}}$, $\#G_{\boldsymbol{\beta}} \leq 1$ then the set $F_{\boldsymbol{\alpha}}$ is countable;
(c) If for all maximal vertices $\boldsymbol{\beta}$ in $\Gamma_{\boldsymbol{\alpha}}$, $\#G_{\boldsymbol{\beta}} = 1$ and $G_{\boldsymbol{\beta}} = \varnothing$ for all other vertices in $\Gamma_{\boldsymbol{\alpha}}$, then the set $F_{\boldsymbol{\alpha}}$ is finite. In this case, $\#F_{\boldsymbol{\alpha}}$ is equal to the sum of all compositions $\prod\limits_{j=1}^{p-1} \#G_{\boldsymbol{\alpha}_j\boldsymbol{\alpha}_{j+1}}$, taken over all chains $\boldsymbol{\alpha} = \boldsymbol{\alpha}_1 \prec \ldots \prec \boldsymbol{\alpha}_p = \boldsymbol{\beta}$, where $\boldsymbol{\beta}$ is maximal in $\Gamma_{\boldsymbol{\alpha}}$;
(d) The set $F_{\boldsymbol{\alpha}}$ is a singleton if and only if $\Gamma_{\boldsymbol{\alpha}}$ is a chain $\boldsymbol{\alpha} = \boldsymbol{\alpha}_1 \prec \ldots \prec \boldsymbol{\alpha}_p$ in which for all $j \leq p - 1$, $\#G_{\boldsymbol{\alpha}_j\boldsymbol{\alpha}_{j+1}} = 1$, $G_{\boldsymbol{\alpha}_j} = \varnothing$ and $\#G_{\boldsymbol{\alpha}_p} = 1$.*

2.3. Intersections of the pieces of a fractal cube K.

To examine the intersections of copies of a single fractal cube K, we consider the intersections of K with itself. In that setting, the sets $F_{\boldsymbol{\alpha}}$ are the intersections of opposite faces $K_{\boldsymbol{\alpha}}$ and $K_{-\boldsymbol{\alpha}}$ of the same fractal cube K, implying several relations between the parameters, containing $\boldsymbol{\alpha}$ and $-\boldsymbol{\alpha}$. If $\boldsymbol{\alpha} = 0$, then $F_0 = K$. If $\boldsymbol{\alpha} \neq 0$, then $F_{-\boldsymbol{\alpha}} = F_{\boldsymbol{\alpha}} - \boldsymbol{\alpha}$. A direct computation shows that, for any $\boldsymbol{\alpha}$, $G_{\boldsymbol{\alpha}} = D \cap (D + (n-1)\boldsymbol{\alpha})$ implies $G_{-\boldsymbol{\alpha}} = G_{\boldsymbol{\alpha}} - (n-1)\boldsymbol{\alpha}$. Similarly, $G_{-\boldsymbol{\alpha}-\boldsymbol{\beta}} = D \cap (D - n\boldsymbol{\alpha} + \boldsymbol{\beta}) = G_{\boldsymbol{\alpha\beta}} - n\boldsymbol{\alpha} + \boldsymbol{\beta}$.

Proposition 2.8. *[8] Let K be a fractal cube with digit set D and let $d_1, d_2 \in D$ and $K_{(d_1)}, K_{(d_2)}$ be copies of K. If $d_2 - d_1 \notin A_k$, then $K_{(d_1)} \cap K_{(d_2)} = \varnothing$. If $d_2 - d_1 = \alpha \in A_k$, then $K_{(d_1)} \cap K_{(d_2)} = \dfrac{F_{\boldsymbol{\alpha}} + d_1}{n}$.*

3. The single intersection property and dendrite criterion.

Definition 3.1. *[7] Let $\mathcal{K} = \{K_i, i \in I = \{1, \dots, m\}\}$ be a finite system of continua in a Hausdorff topological space X. We say that $\mathcal{K}$ has* single intersection property*, if for any $i \neq j \in I$, intersection $P_{ij} = K_i \cap K_j$ consists of at most one point. We call $\mathcal{K}$ a* SIP-set system.

In the context of Definition 3.1, we denote $K = \bigcup\limits_{i\in I} K_i$, $P = \bigcup\limits_{i\neq j} P_{ij}$ and $P_i = \bigcup\limits_{j\in I\setminus\{i\}} P_{ij}$. Taking into account K as the subspace of X provided by the induced topology, we see that the set P_i is the boundary ∂K_i of the set K_i in K, and that its interior is $\dot{K}_i = K_i \setminus P_i$. Observe that for any $i \in I$, $\#\partial K_i \leq (m-1)$.

Definition 3.2. *Let $\mathcal{S} = \{S_1, \dots, S_m\}$ be a system of injective contraction maps on a complete metric space X and K be its attractor. Let $\mathcal{K}(\mathcal{S}) = \{K_1, \dots, K_m\}$. $\mathcal{S}$ is called an SIP system of contractions if the system $\mathcal{K}(\mathcal{S})$ is an SIP set system.*

Applying Theorem 2.7, we get the following single intersection property criterion.

Corollary 3.3. *[8] A fractal cube K has the single intersection property iff the structure graph $\Gamma(\Sigma)$ is a union of chains $0 \prec \boldsymbol{\alpha}_{i1} \prec \dots \prec \boldsymbol{\alpha}_{ip_i}$ for which all $\boldsymbol{\alpha}_{ij}$ are different and such that for all i $\#G_{\boldsymbol{\alpha}_{ip_i}} = 1$ and for all i, j for which $j \leq p_i - 1$, $\#G_{\boldsymbol{\alpha}_{ij}\boldsymbol{\alpha}_{i,j+1}} = 1$, $G_{\boldsymbol{\alpha}_{ij}} = \varnothing$.*

2. For a SIP set system $\mathcal{K}$ (resp. SIP system $\mathcal{S}$) we define its *intersection graph* $\mathcal{G}(\mathcal{K})$ (resp. $\mathcal{G}(\mathcal{S})$) as a bipartite graph $(\mathcal{K}, P; E)$ with parts $\mathcal{K}$ and P, for which an edge $\{K_i, p\} \in E$ iff $p \in K_i$. We call $K_i \in \mathcal{K}$ *white vertices* and $p \in P$ – *black vertices* of the graph Γ. The set $N(K_i)$ of neighbors of any white vertex K_i is P_i, whereas for any black vertex p, $N(p) = \{K_i : p \in K_i\}$. Each $p \in P$ is the intersection point of at least two of the sets K_i, therefore, $\deg(p) \geq 2$.

Theorem 3.4. *[1, 7]. Let $\mathcal{S}$ be a system of injective contraction maps in a complete metric space X, which has the single intersection property. The attractor K of the system $\mathcal{S}$ is a dendrite if and only if the intersection graph $\mathcal{G}(\mathcal{S})$ is a tree.*

Theorem 3.5. *If a fractal k-cube $K = \dfrac{K+D}{n}$ has the single intersection property and its intersection graph $\mathcal{G}$ is a tree, then K is a dendrite.*

3.1. Finding black ramification points in the graph $\mathcal{G}$.

Theorem 3.6. *Let $K = \dfrac{K+D}{n}$ be a fractal cube which has the single intersection property.*

If there are $\boldsymbol{\alpha}, \boldsymbol{\beta} \succ 0$ such that $F_{\boldsymbol{\alpha}} = F_{\boldsymbol{\beta}} \neq \varnothing$, then for any triple $d, d+\boldsymbol{\alpha}, d+\boldsymbol{\beta} \in D$ the copies $K_{(d)}$, $K_{(d+\boldsymbol{\alpha})}$ and $K_{(d+\boldsymbol{\beta})}$ intersect in a single point $x = \dfrac{F_{\boldsymbol{\alpha}}+d}{n} = \dfrac{F_{\boldsymbol{\beta}}+d}{n}$.

Proof. It is clear that $K_{(d)} \cap K_{(d+\boldsymbol{\alpha})} = \dfrac{F_{\boldsymbol{\alpha}}+d}{n}$ и $K_{(d)} \cap K_{(d+\boldsymbol{\beta})} = \dfrac{F_{\boldsymbol{\beta}}+d}{n}$. The equality $F_{\boldsymbol{\alpha}} = F_{\boldsymbol{\beta}}$, implies

$$K_{(d)} \cap K_{(d+\boldsymbol{\alpha})} = K_{(d)} \cap K_{(d+\boldsymbol{\beta})} = \frac{F_{\boldsymbol{\beta}}+d}{n} = \frac{F_{\boldsymbol{\alpha}}+d}{n} = \{x\},$$

therefore $K_{(d)} \cap K_{(d+\boldsymbol{\alpha})} \cap K_{(d+\boldsymbol{\beta})} = \{x\}$.

Therefore the white vertices corresponding to $K_{(d)}, K_{(d+\boldsymbol{\alpha})}, K_{(d+\boldsymbol{\beta})}$ are connected to the same single black vertex p corresponding to the point x.

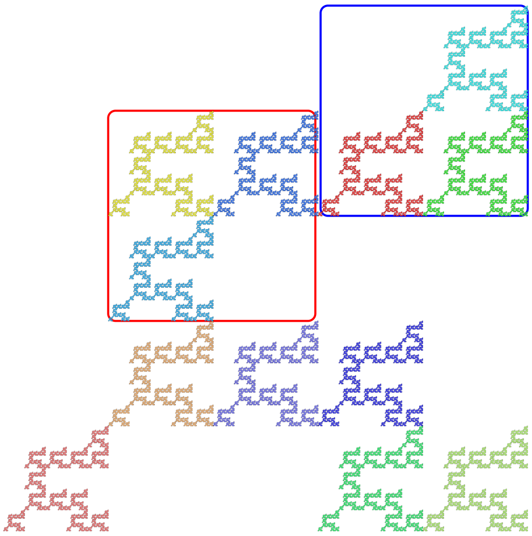
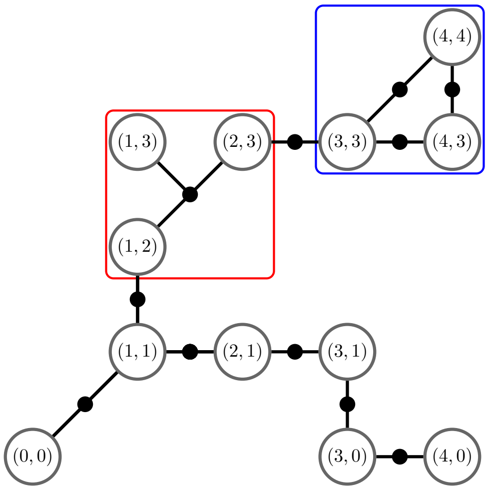

Figure 3: A triple of copies with unique intersection point and a triple of copies forming a cycle.

On the other hand, if $d, d-\boldsymbol{\alpha}, d-\boldsymbol{\beta} \in D$ then the intersection points are different.

$$K_{(d)} \cap K_{(d-\boldsymbol{\alpha})} = \frac{F_{\boldsymbol{\alpha}} + (d-\boldsymbol{\alpha})}{n};\ K_{(d)} \cap K_{(d-\boldsymbol{\beta})} = \frac{F_{\boldsymbol{\alpha}} + (d-\boldsymbol{\beta})}{n};$$

$$K_{(d-\boldsymbol{\alpha})} \cap K_{(d-\boldsymbol{\beta})} = \frac{F_{\boldsymbol{\alpha}-\boldsymbol{\beta}} + (d-\boldsymbol{\alpha})}{n} = \frac{F_{\boldsymbol{\beta}-\boldsymbol{\alpha}} + (d-\boldsymbol{\beta})}{n}$$

Therefore the white vertices corresponding to $K_{(d)}, K_{(d+\boldsymbol{\alpha})}, K_{(d+\boldsymbol{\beta})}$ form a cycle consisting of 3 white and 3 black points in the graph $\mathcal{G}$. □

Corollary 3.7. *Let $B = \{d_1, ..., d_m\}$ be a subset of D which satisfies the condition: For any $d_i, d_j, d_k \in B$,*

$$(d_j - d_i), (d_k - d_i) \in A \setminus 0 \text{ and } F_{d_j - d_i} = F_{d_k - d_i} \neq \varnothing.$$

Then there is a point $x \in K$ such that for any $d_i, d_j \in B$, $K_{(d_i)} \cap K_{(d_j)} = \{x\}$, and x corresponds to a black vertex of order m in $\mathcal{G}$.

4. How to test the dendrite property for a fractal cube.

Let K be a fractal k-cube. To check the dendrite property for K one should perform the following steps:

1. Find all the sets $G_{\boldsymbol{\alpha}}, G_{\boldsymbol{\alpha\beta}}$ for the system $\Sigma = \Sigma(K, K)$, and write the system Σ. Following Definition 2.5 and Lemma 2.6, eliminate all vanishing vertices and edges and construct the graph Γ_Σ.

2. Using the Corollary 3.3 check the single intersection property for the cube K If it fails, K is not a dendrite.

3. Construct the bipartite intersection graph for the fractal cube K, connecting the intersecting copies with their intersection point by the edges. Be aware of multiple points, mentioned in the Corollary 3.7. If this graph is a tree, then K is a dendrite.

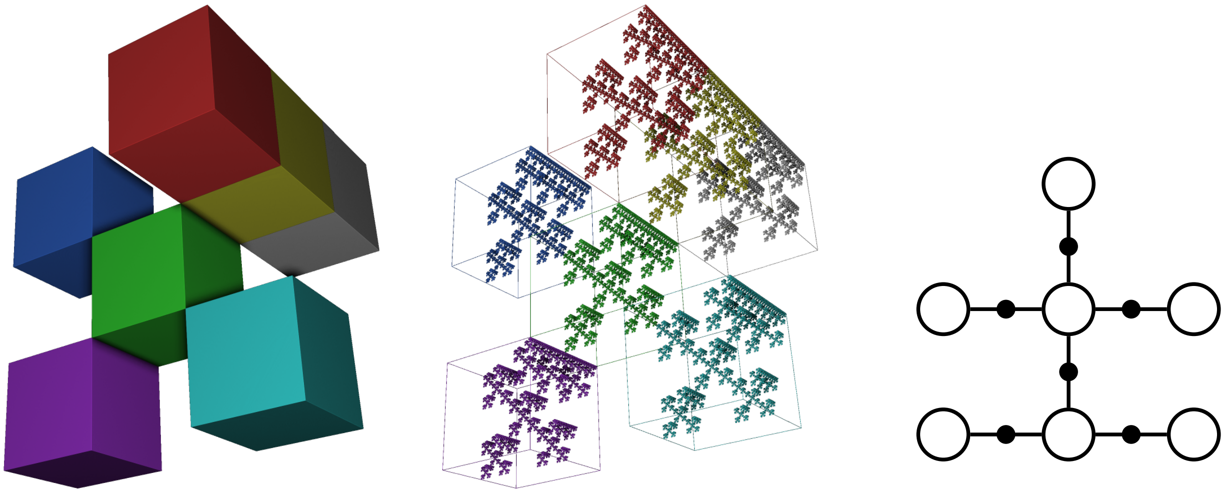

Figure 4: A fractal cube dendrite and its intersection graph.[9]

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